The propagation of systematic uncertainties in the intensity calibration for HIFI

Version 2.1

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1 Introduction

The HIFI intensity calibration framework (ALMA Memo 442.1) provides all equations that were used or are usable for the translation of raw HIFI spectra into intensity-calibrated data by the pipeline software. It also includes equations for the error propagation of statistic and systematic uncertainties into the data final product. Sect. 4.3.1 provided all equations for the systematic uncertainties, excluding standing waves.

However, at the time of the development of the framework, little was known about the nature of the standing waves in the instrument and very optimistic assumptions were made with respect to our ability to characterize the standing waves and the coupling to the telescope and to include all possible corrections in the calibration pipeline. As we did not manage to quantify all the terms from the framework, only part of the document has been implemented in the pipeline. In particular the "off calibration" (Sect. 3) proposed in the framework has never been used. Consequently the error propagation has to be reevaluated for the equations that are eventually implemented. Moreover, the framework document computed the error terms only explicitly for the extreme cases of band 1 and band 7b observations, but not for the general case that should be implemented for the error propagation in the pipeline.

Here, the equations for the propagation of uncertainties are reevaluated using the following assumptions:

- The pipeline only provides temperatures on the scale of antenna temperatures, i.e. all uncertainties in terms of the beam efficiencies $\eta_{\rm sf}$ are neglected.
- The coupling to the telescope was never accurately characterized by summing all the OFF spectra, i.e. we have no information about the actual forward efficiency $\eta_{\rm l}$ and the modulation of that factor by standing waves. Without any information we have to ignore this term and its uncertainties here.
- For practical reasons the sideband ratio of HIFI has been defined as the one for the upper sideband $G_{\rm usb} = \gamma_{\rm usb}/(\gamma_{\rm lsb} + \gamma_{\rm usb})$.

- We now know that the main sources of standing waves in HIFI are standing waves in the cavities to the thermal loads and the diplexer standing wave in the HIFI bands using a diplexer to mix sky signal with the LO signal. They can be separated based on their characteristic period.
- The standing waves in both sidebands are only measured in their combined impact on continuum signals. We have not been able to separate them except for a few lines.
- The HIFI pipeline will not be changed with respect to the correction for standing waves, i.e. sll standing waves are only considered as uncertainties.

In a first step we rephrase the calibration equations to take the current description of standing waves into account. In a second step we obtain the derivatives of the equations to obtain the individual error propagation terms.

For the sake of clarity, we phrase all general equations and those for the continuum calibration using the subscripts "usb" and "lsb" for the two sidebands. For the single-sideband calibration of a line we use the index "ssb" for the signal sideband and "isb" for the image sideband. Depending on whether the line is in the upper or lower sideband, they have to be replaced by the corresponding "usb" or "ssb" index, impying e.g. that $G_{\rm ssb} = G_{\rm usb}$ and $G_{\rm isb} = 1 - G_{\rm usb}$ for the line in the upper sideband and $G_{\rm ssb} = 1 - G_{\rm usb}$ and $G_{\rm isb} = G_{\rm usb}$ for the line in the lower sideband.

2 Standing wave description

Standing waves are described as a modulation of the coupling factor to the individual sources. Even if they are not known separately for the two sidebands, we have to define them separately to match to the calibration equations

$$\begin{aligned} \gamma_{\rm usb} &= w_{\rm usb} G_{\rm usb} \gamma_{\rm rec} \\ \gamma_{\rm lsb} &= w_{\rm lsb}(1 - G_{\rm usb})\gamma_{\rm rec} \end{aligned} \tag{1}$$

This can be substituted in the equations of Sect. 2 of the framework document. A standing wave factor $w = 1$ represents the lack of standing waves, a superposition of sinusoidal variations around unity reflect the full behavior.

For the continuum calibration only the sum of both terms count, i.e. only the factor $[w_{\rm lsb} + (w_{\rm usb} - w_{\rm lsb})G_{\rm usb}]\gamma_{\rm rec}$ appears in the calibration equations (see Eq. 11). Hence, we do not know the individual single sideband standing waves $w_{\rm usb}$ and $w_{\rm ssb}$, but only the standing wave measured on the combined continuum

$$W = G_{\rm usb}(w_{\rm usb} - w_{\rm lsb}) + w_{\rm lsb} \tag{2}$$

We can describe the standing waves as a deviation from the coupling factor of unity, $W = 1 + \hat{W}$, $w_{\rm usb} = 1 + \hat{w}_{\rm usb}$, $w_{\rm lsb} = 1 + \hat{w}_{\rm lsb}$ and get

$$\hat{W} = G_{\rm usb}(\hat{w}_{\rm usb} - \hat{w}_{\rm lsb}) + \hat{w}_{\rm lsb} \tag{3}$$

If we assume that the amplitude of the standing wave in both sidebands is approximately equal and the sideband ratio $G_{\rm usb}$ is approximately 0.5, the sum $\hat{W} = 1/2(\hat{w}_{\rm usb} +$

$\hat{w}_{\rm lsb}$) can take any value from total cancellation of the standing waves up to a sum that provides the same amplitude as the single-sideband standing waves in spite of the factor $G_{\rm usb} \approx 0.5$, i.e. $\hat{W}$ falls between 0 and $\hat{w}_{\rm ssb}$. On average the continuum standing wave has half the amplitude of the single sideband standing waves, or vice versa, the single-sideband standing wave has twice the amplitude as measured for the continuum sources.

We can also write this as a relation of the amplitudes of the distortion between single sideband and double sideband standing waves

$$\begin{aligned} \hat{w}_{\rm usb} &= \frac{\hat{W}}{G_{\rm usb}} - \frac{1-G_{\rm usb}}{G_{\rm usb}} \times \hat{w}_{\rm lsb} \\ \hat{w}_{\rm lsb} &= \frac{\hat{W}}{1-G_{\rm usb}} - \frac{G_{\rm usb}}{1-G_{\rm usb}} \times \hat{w}_{\rm usb} \end{aligned} \tag{4}$$

Furthermore, we have to distinguish the standing wave contributions from the loads and from the diplexer. They combine to the total standing wave distortion

$$\begin{aligned} \hat{w}_{\rm usb} &= \hat{w}_{\rm usb,loads} + \hat{w}_{\rm usb,dipl} \\ \hat{w}_{\rm lsb} &= \hat{w}_{\rm lsb,loads} + \hat{w}_{\rm lsb,dipl} \\ \hat{W} &= G_{\rm usb}(\hat{w}_{\rm usb,loads} + \hat{w}_{\rm usb,dipl}) + (1-G_{\rm usb})(\hat{w}_{\rm lsb,loads} + \hat{w}_{\rm lsb,dipl}) \end{aligned} \tag{5}$$

In the HIFI beam splitter bands, we can assume that the standing waves to the loads are the only standing waves in the system, i.e. that there are no standing waves in the system for sky observations. Then the only error term comes from the modulated response in the load measurement expressed through $\hat{w}_{\rm usb} = \hat{w}_{\rm usb,loads}$ and $\hat{w}_{\rm lsb} = \hat{w}_{\rm lsb,loads}$. In the diplexer bands, a fraction of the continuum standing waves stems from the diplexer, as described by Eq. (5). Only this fraction is also applicable in the path to the sky, i.e. there we have

$$\hat{W}_{\rm sky} = G_{\rm usb}(\hat{w}_{\rm usb,dipl} - \hat{w}_{\rm lsb,dipl}) + \hat{w}_{\rm lsb,dipl} \tag{6}$$

3 Load measurement

3.1 Change of the bandpass measurement by standing waves

When including the standing waves into the coupling factors to the loads in Eqs. (7) and (8) of the framework document, the effectively measured bandpass has an additional standing wave term so that in Eq. (9) from the framework the effective thermal radiation fields detected by the receiver changes relative to Eq. (12) from the framework into

$$J_{\rm eff} = w_{\rm usb} G_{\rm usb} J_{\rm usb} + w_{\rm lsb}(1-G_{\rm usb}) J_{\rm lsb} \tag{7}$$

Using the Planck function for J (Eq. (4) from framework) and the different intensity levels in both sidebands (Eq. (5) from framework)

$$J_{\rm USB} - J_{\rm LO} = J_{\rm LO} - J_{\rm LSB} = b\nu_{\rm IF} \times J_{\rm LO} \tag{8}$$

we can compute the sideband difference

$$b(\nu_{\rm LO}, T) = \frac{1}{\nu_{\rm LO}}(3 - e_{\rm P}) = \frac{1}{\nu_{\rm LO}}\left(3 - p \times \frac{\exp(p)}{\exp(p) - 1}\right) \tag{9}$$

with p being the Planck exponent

$$p = \frac{h\nu_{\rm LO}}{kT} \,. \tag{10}$$

This leads to the general expression for the effective radiation temperature of

$$\begin{aligned} J_{\rm eff} &= [w_{\rm lsb} + (w_{\rm usb} - w_{\rm lsb})G_{\rm usb}]\, J_{\rm LO} \\ &\quad - [w_{\rm lsb} - (w_{\rm usb} + w_{\rm lsb})G_{\rm usb}]\, J_{\rm LO} \times b\nu_{\rm IF} \end{aligned} \tag{11}$$

However, as it is not intended to change the definition of the receiver bandpass in the pipeline we can turn the main standing wave term into a multiplicative correction. The bandpass from the pipeline is then:

$$\gamma'_{\rm rec} = [G_{\rm usb}w_{\rm usb} + (1 - G_{\rm usb})w_{\rm lsb}]\gamma_{\rm rec} = W\gamma_{\rm rec} = \frac{c_{\rm hot} - c_{\rm cold}}{(\eta_{\rm h} + \eta_{\rm c} - 1)(J'_{\rm h,eff} - J'_{\rm c,eff})} \tag{12}$$

(see Eq. (9) from framework) where $J'_{\rm eff}$ deviates from the used $J_{\rm eff}$ from Eq. (12) in the framework only by a minor correction

$$J_{\rm eff,framework} = J_{\rm LO}\,(1 + (2G_{\rm usb} - 1)b\nu_{\rm IF}) \tag{13}$$

$$\begin{aligned} J'_{\rm eff} &= J_{\rm LO}\left(1 + \frac{G_{\rm usb}(w_{\rm usb} + w_{\rm lsb}) - w_{\rm lsb}}{G_{\rm usb}(w_{\rm usb} - w_{\rm lsb}) + w_{\rm lsb}} b\nu_{\rm IF}\right) \\ &= J_{\rm LO}\left(1 + \frac{(2G_{\rm usb} - 1) + \hat{W} + 2G_{\rm usb}\hat{w}_{\rm usb}}{1 + \hat{W}} b\nu_{\rm IF}\right) \end{aligned} \tag{14}$$

$$= J_{\rm LO}\left(1 + \frac{(2G_{\rm usb} - 1) + \hat{W} + 2(1 - G_{\rm usb})\hat{w}_{\rm lsb}}{1 + \hat{W}} b\nu_{\rm IF}\right) \tag{15}$$

where we used Eqs. (4) to express the single sideband standing waves by the measured continuum standing wave $\hat{W}$. The ratio in Eqs. (14) and 815) describes the change of the continuum sideband imbalance by standing waves. As it is proportional to the continuum slope between the two sidebands $b(\nu_{\rm LO}, T)$ the difference between the "correct" definition of the intensity detected by HIFI and the definition used in the pipeline is small.

3.2 Error contributions

3.2.1 Standing wave contribution

As it is not foreseen that the pipeline uses the new full expression for the effective temperature of hot and cold load in the determination of the bandbass, the standing wave contribution in $J'_{\rm eff}$ has to be added as a general error of $\gamma'_{\rm rec}$ given by the amplitude of the sum of the standing waves.

If we assume that the standing wave amplitude is small, $\hat{W} \ll 1$, we can ignore the denominator in Eq. (14) and obtain the error of $J'_{\rm eff}$ from

$$J'_{\rm eff} = J_{\rm LO}\,(1 + (2G_{\rm usb} - 1 + \hat{W} + 2G_{\rm usb}\hat{w}_{\rm usb})b\nu_{\rm IF}) \tag{16}$$

The resulting error in the bandbass then follows from

$$\delta\gamma'_{\rm rec} = \left(\frac{\partial\gamma'_{\rm rec}}{\partial J_{\rm eff,h}}\frac{\partial J_{\rm eff,h}}{\partial W} + \frac{\partial\gamma'_{\rm rec}}{\partial J_{\rm eff,c}}\frac{\partial J_{\rm eff,c}}{\partial W}\right) \times (\hat{W} + 2G_{\rm usb}\hat{w}_{\rm usb}) \tag{17}$$

$$\approx -\gamma'_{\rm rec}(\hat{W} + 2G_{\rm usb}\hat{w}_{\rm usb})\nu_{\rm IF} \times \frac{J_{\rm eff,h}b(\nu_{\rm LO}, T_{\rm h}) - J_{\rm eff,c}b(\nu_{\rm LO}, T_{\rm c})}{J_{\rm eff,h} - J_{\rm eff,c}} \tag{18}$$

$$\approx -\gamma'_{\rm rec}(\hat{W} + 2G_{\rm usb}\hat{w}_{\rm usb}) \times \Delta_{\rm sb} = \gamma'_{\rm rec}(\hat{W} + 2G_{\rm ssb}\hat{w}_{\rm ssb}) \times \Delta_{\rm sb} \tag{19}$$

using the expression for the intrinsic sideband imbalance of the continuum difference radiation

$$\Delta_{\rm sb} = \nu_{\rm IF} \times \frac{J_{\rm LO}(T_{\rm h})b(\nu_{\rm LO}, T_{\rm h}) - J_{\rm LO}(T_{\rm c})b(\nu_{\rm LO}, T_{\rm c})}{J_{\rm LO}(T_{\rm h}) - J_{\rm LO}(T_{\rm c})} \tag{20}$$

and Eq. (15) to substitute the $2G_{\rm usb}\hat{w}_{\rm usb}$ by the general term $2G_{\rm ssb}\hat{w}_{\rm ssb}$ as needed in Sect. 4.

3.2.2 Other systematic uncertainties

Other systematic errors in the load calibration can be estimated straightforward from the uncertainty of the sideband ratio $G_{\rm ssb}$, the cold and hot load temperature, and the cold and hot load coupling coefficients $\eta_{\rm c}$ and $\eta_{\rm h}$. If we ignore the second-order standing wave terms and assume that the load coupling coefficients are close to unity we obtain

$$\delta\gamma'_{\rm rec} = \frac{\partial\gamma'_{\rm rec}}{\partial\eta_{\rm h}}\delta\eta_{\rm h} + \frac{\partial\gamma'_{\rm rec}}{\partial\eta_{\rm c}}\delta\eta_{\rm c} + \frac{\partial\gamma'_{\rm rec}}{\partial J_{\rm eff,h}}\delta J_{\rm eff,h} + \frac{\partial\gamma'_{\rm rec}}{\partial J_{\rm eff,c}}\delta J_{\rm eff,c} \tag{21}$$

$$\approx -\gamma'_{\rm rec}\left(\delta\eta_{\rm h} + \delta\eta_{\rm c} + \frac{\delta J_{\rm eff,h} - \delta J_{\rm eff,c}}{J_{\rm eff,h} - J_{\rm eff,c}}\right) . \tag{22}$$

For the error estimate all values of $J_{\rm eff}$ van be approximated by the corresponding $J_{\rm LO}$ and we obtain

$$\frac{\delta J_{\rm eff,h} - \delta J_{\rm eff,c}}{J_{\rm eff,h} - J_{\rm eff,c}} = 2\delta G_{\rm usb}\Delta_{\rm sb} + f\left(\frac{\delta T_{\rm h}}{T_{\rm h}}, \frac{\delta T_{\rm c}}{T_{\rm c}}\right) \tag{23}$$

where we use the notation

$$f\left(\frac{\delta T_{\rm h}}{T_{\rm h}}, \frac{\delta T_{\rm c}}{T_{\rm c}}\right) = \frac{J_{\rm LO}(T_{\rm h})e_{\rm P}(\nu_{\rm LO}, T_{\rm h})\delta T_{\rm h}/T_{\rm h} - J_{\rm LO}(T_{\rm c})e_{\rm P}(\nu_{\rm LO}, T_{\rm c})\delta T_{\rm c}/T_{\rm c}}{J_{\rm LO}(T_{\rm h}) - J_{\rm LO}(T_{\rm c})} \tag{24}$$

with $e_{\rm P}(\nu_{\rm LO}, T)$ defined in Eq. (9). For describing the error of a full spectrum, we have to decide which value of $\nu_{\rm IF}$ to use. For a typical error it may be reasonable to use the IF frequency of the band center, i.e. 6 or 3.6 GHz for SIS and HEB bands respectively. If we want to quantify the maximum error, the band edge, i.e. 8 or 4.8 GHz have to be used for SIS and HEB bands, respectively.

Eqs. (19) and (22) describe the full error propagation assuming that uncertainties are know including their sign. This allows for the exact treatment of mutually dependent errors, quantities with asymmetric errors bars and the error propagation in the final calibration equation if the same uncertainty enters in different terms.

Practically it is not clear to what accuracy the uncertainty of individual input parameters can be determined. If we have no information about mutual correlations and signed errors, the three individual terms in Eq. (22) should be added quadratically instead of linearly. This would apply to the actual uncertainty of the bandpass. However, we will continue with the full, signed errors here because of a partial cancellation of the sideband ratio and standing wave term in the uncertainty of the final calibration equation in Sect. 4.

4 The calibration equation

The calibration equation for multiplicative standing waves is given in Eq. (40) of the framework document. Ignoring the beam efficiency there and including the standing wave term, it reads as

$$J_S - J_R = \frac{c_S - c_R}{\eta_l \gamma_{rec} G_{ssb} w_{sky,ssb}} \tag{25}$$

$$= \frac{(c_S - c_R) W}{\eta_l \gamma'_{rec} G_{ssb} w_{sky,ssb}} \tag{26}$$

As the pipeline does not use the full calibration equation, we have to consider the additional standing wave terms W and $w_{sky,ssb}$ as error terms.

4.1 Standing waves

We can distinguish three different cases for the standing wave contribution. The most simple case occurs in the beam splitter bands. According to the premises made in Sect. 1, there should be no additional standing wave in the sky path, i.e. the standing wave term in the denominator $w_{sky,ssb} = 1$. The standing-wave modification both for lines and continuum is done through the double-sideband standing wave term W in the load measurement translating γ'_{rec} into γ_{rec} The standing wave error in the calibration is directly given by the amplitude of the measured standing wave distortion $\delta W = \hat{W}_{loads}$.

The second case is given for observations aim at the continuum in the diplexer bands. There the sky standing wave is only a fraction of the standing wave seen in the load measurent. We can write Eq. (26) as

$$J_{S,cont} - J_{R,cont} = \frac{(c_S - c_R)(1 + \hat{W}_{loads} + \hat{W}_{dipl})}{\eta_l \gamma'_{rec} (1 + \hat{W}_{sky})} \tag{27}$$

As we assume that the standing wave amplitude is small, we can approximate

$$\frac{1 + \hat{W}_{loads} + \hat{W}_{dipl}}{1 + \hat{W}_{sky}} \approx 1 + \hat{W}_{loads} + \hat{W}_{dipl} - \hat{W}_{sky} \tag{28}$$

Without diplexer standing wave, $\hat{W}_{dipl} = 0$ and $\hat{W}_{sky} = 0$, we reproduce the first case for the continuum. If the diplexer standing wave is stable, $\hat{W}_{dipl} = \hat{W}_{sky}$ the diplexer standing wave is fully calibrated out through the load measurement, i.e. only $\hat{W}_{loads}$ remains. If we have a pure diplexer standing wave, i.e. $\hat{W}_{loads} = 0$, all standing waves are calibrated out through the load calibration. For any temporal variation of the diplexer standing wave we need to keep both the diplexer standing wave for the load measurement $\hat{W}_{\mathrm{dipl}}$ and the one for the astronomical observbation, $\hat{W}_{\mathrm{sky}} = 0$.

The third and most complex case refers to the calibration of line observations in diplexer bands where we do not know the single sideband standing waves, so that we can only guess the amplitude of $\hat{w}_{\mathrm{sky,ssb}}$ from Eq. (4). Using the same linear approximation as in Eq. (28) we obtain

$$J_{\mathrm{S,lines}} - J_{\mathrm{R,lines}} = \frac{(c_{\mathrm{S}} - c_{\mathrm{R}})(1 + \hat{W} - \hat{w}_{\mathrm{ssb,sky}})}{\eta_{\mathrm{l}} \gamma'_{\mathrm{rec}} G_{\mathrm{ssb}}} \tag{29}$$

The equation implicitly contains the two easier cases. For the beam splitter bands, we have no $\hat{w}_{\mathrm{ssb,sky}}$ contribution and $\hat{W}$ is determined by the standing waves to the loads. For the continuum, we drop the sideband ratio in the denominator and we can use $\hat{W} = \hat{W}_{\mathrm{loads}}$ as the only term if the diplexer standing wave is stable.

In the general case and if the diplexer standing wave is fluctuating between load measurement and astronomical observation, i.e. $\hat{w}_{\mathrm{ssb,sky}} \neq \hat{w}_{\mathrm{ssb,dipl}}$ we obtain only a partial cancellation of the terms in Eq. (5) or even an amplification.

As discussed in Sect. 2, the pipeline does not correct for $\hat{W}$ or $\hat{w}_{\mathrm{ssb}}$. Hence, they always provide a 100 % relative error in their contribution to the error of the calibrated spectra. Following Eq. 3, we can approximate $\hat{w}_{\mathrm{ssb,dipl}} = \hat{W}_{\mathrm{dipl}} / G_{\mathrm{ssb}}$.

4.2 Calibration uncertainties

As causes for systematic errors the framework document already discussed uncertainties in the sideband ratio G_{ssb} and in the receiver bandpass $\gamma^{\mathrm{l}}_{\mathrm{rec}}$. However, as the pipeline did not implement the "off calibration" (Sect. 3) of the framework document, the error propagation has changed because of missing cancellations through the η_{l} term. On top of this we have to add the uncertainty from the standing waves. For the mathematical description it is sufficient to consider the calibration equation (29). It contains the other cases. For the continuum calibration we can just substitute G_{ssb} by unity, and for the beam-splitter case we just have $w_{\mathrm{ssb,dipl}} = 0$.

Because of the simple linear dependence for the three quantities we obtain

$$\frac{\delta(J_{\mathrm{S}} - J_{\mathrm{R}})}{J_{\mathrm{S}} - J_{\mathrm{R}}} \approx -\frac{\delta \gamma'_{\mathrm{rec}}}{\gamma'_{\mathrm{rec}}} - \frac{\delta G_{\mathrm{ssb}}}{G_{\mathrm{ssb}}} + \hat{W} - \hat{w}_{\mathrm{ssb,sky}} \tag{30}$$

and from Eqs. (23) and (19) we have

$$\frac{\delta \gamma'_{\mathrm{rec}}}{\gamma'_{\mathrm{rec}}} = (\hat{W} + 2 G_{\mathrm{ssb}} \hat{w}_{\mathrm{ssb}} + 2 \delta G_{\mathrm{usb}}) \Delta_{\mathrm{sb}} + \delta \eta_{\mathrm{h}} + \delta \eta_{\mathrm{c}} + f\left(\frac{\delta T_{\mathrm{h}}}{T_{\mathrm{h}}}, \frac{\delta T_{\mathrm{c}}}{T_{\mathrm{c}}}\right) \tag{31}$$

where $f\left(\delta T_{\mathrm{h}} / T_{\mathrm{h}}, \delta T_{\mathrm{c}} / T_{\mathrm{c}}\right)$ is defined in Eq. 24.

The terms for the standing wave uncertainty and the sideband ratio appear twice leading to a linear combination, i.e. a mutual reduction or amplification. We can express the combination of $\hat{W}$ and $\hat{w}_{\mathrm{ssb}}$ through Eq. (4). From the sideband imbalance of the continuum difference (Eq. 20) we have to distinguish different signs for USB and LSB observations. Furthermore, we can approximate a sideband ratio of about 1/2 for this term and obtain

$$\begin{aligned}\frac{\delta(J_\mathrm{S}-J_\mathrm{R})}{J_\mathrm{S}-J_\mathrm{R}} \approx\ & \hat{W}-\hat{w}_\mathrm{ssb,sky} \\ & -\Delta_\mathrm{sb}(3\hat{W}-2G_\mathrm{isb}\hat{w}_\mathrm{isb}) \\ & -(1\pm\Delta_\mathrm{sb})\frac{\delta G_\mathrm{ssb}}{G_\mathrm{ssb}} \\ & -\delta\eta_\mathrm{h}-\delta\eta_\mathrm{c}-f\left(\frac{\delta T_\mathrm{h}}{T_\mathrm{h}},\frac{\delta T_\mathrm{c}}{T_\mathrm{c}}\right)\end{aligned} \tag{32}$$

where the $\pm$ indicates a positive sign if we observe in upper sideband, ssb=usb, and a negative sign in the lower sideband, ssb=lsb.

The sky observations only see the diplexer standing wave. If the diplexer standing wave is stable, one could get one more cancelation through $\hat{w}_\mathrm{ssb,sky}=\hat{w}_\mathrm{ssb,dipl}$ using the split of the standing wave description into the contribution from the loads and from the diplexer

$$\hat{W}-\hat{w}_\mathrm{ssb,sky}=\hat{W}_\mathrm{loads}+\hat{W}_\mathrm{dipl}-\hat{w}_\mathrm{ssb,dipl}\,. \tag{33}$$

However, as we do not know the individual sideband contribution, the overall uncertainty of $\hat{W}_\mathrm{dipl}-\hat{w}_\mathrm{ssb,dipl}$ is as big as that of $\hat{W}_\mathrm{dipl}+\hat{w}_\mathrm{ssb,dipl}$, i.e. we only obtain a practical cancellation for continuum observations where $\hat{w}_\mathrm{ssb,dipl}$ is substituted by $\hat{W}_\mathrm{ssb,dipl}$.

If we assume that all uncertainties are statistically independent, as well as the standing waves from the loads and the diplexer and the standing wave contributions from both sidebands, we can replace the sum of the individual uncertainties in Eq. (32) by a quadratic sum. Furthermore, we can estimate the uncertainty of the single sideband standing waves from the continuum standing wave as discussed in Sect. 2 as $\hat{w}_\mathrm{ssb,sky}\approx\hat{w}_\mathrm{ssb,dipl}\approx\hat{W}_\mathrm{dipl}/G_\mathrm{ssb}$ and obtain

$$\begin{aligned}\frac{\delta(J_\mathrm{S}-J_\mathrm{R})}{J_\mathrm{S}-J_\mathrm{R}} \approx\ & \Bigg\{\left[(1-3\Delta_\mathrm{sb})^2+(2\Delta_\mathrm{sb})^2\right]\hat{W}^2+\left(\frac{\hat{W}_\mathrm{dipl}}{G_\mathrm{ssb}}\right)^2 \\ & +\left[(1\pm\Delta_\mathrm{sb})\frac{\delta G_\mathrm{ssb}}{G_\mathrm{ssb}}\right]^2+\delta\eta_\mathrm{h}^2+\delta\eta_\mathrm{c}^2+f^2\left(\frac{\delta T_\mathrm{h}}{T_\mathrm{h}},\frac{\delta T_\mathrm{c}}{T_\mathrm{c}}\right)\Bigg\}^{1/2} \qquad (34) \\ \approx\ & \Bigg\{(1-6\Delta_\mathrm{sb})\hat{W}_\mathrm{loads}^2+\left(\frac{1}{G_\mathrm{ssb}^2}+1-6\Delta_\mathrm{sb}\right)\hat{W}_\mathrm{dipl}^2 \\ & +\left[(1\pm\Delta_\mathrm{sb})\frac{\delta G_\mathrm{ssb}}{G_\mathrm{ssb}}\right]^2+\delta\eta_\mathrm{h}^2+\delta\eta_\mathrm{c}^2+f^2\left(\frac{\delta T_\mathrm{h}}{T_\mathrm{h}},\frac{\delta T_\mathrm{c}}{T_\mathrm{c}}\right)\Bigg\}^{1/2} \qquad (35)\end{aligned}$$

where it is assumed that the uncertainty from the temperature is obtained from Eq. (24). This implies that the temperature errors of the cold and hot load are correlated as it would be the case for calibration uncertainties of the devices, readout uncertainties due to power fluctuations or similar systematic effects. However, if the temperature readouts suffer from random fluctuations, the sum in Eq. (24) also has to be replaced by a quadratic sum, i.e. in that case we have to use

$$f\left(\frac{\delta T_\mathrm{h}}{T_\mathrm{h}},\frac{\delta T_\mathrm{c}}{T_\mathrm{c}}\right)=\sqrt{\left[\frac{J_\mathrm{LO}(T_\mathrm{h})e_\mathrm{P}(\nu_\mathrm{LO},T_\mathrm{h})}{J_\mathrm{LO}(T_\mathrm{h})-J_\mathrm{LO}(T_\mathrm{c})}\frac{\delta T_\mathrm{h}}{T_\mathrm{h}}\right]^2+\left[\frac{J_\mathrm{LO}(T_\mathrm{c})e_\mathrm{P}(\nu_\mathrm{LO},T_\mathrm{c})}{J_\mathrm{LO}(T_\mathrm{h})-J_\mathrm{LO}(T_\mathrm{c})}\frac{\delta T_\mathrm{c}}{T_\mathrm{c}}\right]^2} \tag{36}$$

5 Summary

The description of the standing waves as a simple multiplicative term to the calibration equation is well justified when going through all details of the calibration equation. In case of a negligible diplexer standing wave, it is even exact, except for the neglect of the small change of the continuum sideband imbalance by load standing waves.

Unfortunately, it is not foreseen to implement a standing wave model directly into the pipeline. Hence, we have to use the amplitude of all standing waves as 100 % errors. Moreover, we do not know the split of the measured double-sideband standing wave into single sideband standing waves that enter the calibration equation. This leads to an amplification by typically a factor two.

When knowing all calibration uncertainties and the standing wave amplitudes for the standing waves towards the thermal loads and within the diplexer, the error propagation is given by Eq. (35) combined with Eq. (24) or (36), depending on the nature of the uncertainty of the temperature readouts. The equation describes the general case of the uncertainty of the line calibration. For the continuum calibration, the error is reduced due to the missing amplification of the standing wave by the $1/G_{\rm ssb}$ factor. If the diplexer standing wave is stable, it is even completely cancelled out in the error budget there.