Question Set 1 — Probability and Bayes

Question 1

Here are four equations relating some probabilities to other probabilities.

$$P(A \vee B) = P(A) + P(B) \tag{1}$$

$$P(X, Y \mid Z) = P(X \mid Z)P(Y \mid X, Z) \tag{2}$$

$$P(a = 3, b = 4 \mid c = 5) = \frac{P(a = 3, b = 4)P(c = 5 \mid a = 3, b = 4)}{P(c = 5)} \tag{3}$$

$$P(x = 3) = P(x = 3 \mid y = 4)P(y = 4) + P(x = 3 \mid y = 5)P(y = 5). \tag{4}$$

For each of the four equations,

(a) Which rule(s) of probability theory the equation is based upon?;

(b) Is the equation completely general? If not, what extra assumptions have been made (e.g. are some of the propositions independent, or perhaps mutually exclusive?)

Question 2

Steve is on trial for a murder. Based on all the evidence presented in court so far, the jury has narrowed down the set of possibilities to the following three mutually exclusive hypotheses:

Hypothesis	Definition	Prior Probability
S	**S**teve did it	0.5
M	Steve's brother **M**ike did it	0.3
U	Someone **U**nrelated to Steve did it	0.2

Suppose also that Steve wears size 10 shoes all the time, Mike wears size 10 shoes about half of the time, and about 10% of people in general wear men's size 10 shoes.

The murderer's shoeprint is found at the scene, and it is of a men's size 10 shoe. Let D, for data, be the statement that the shoeprint was a men's size 10.

(a) Find the three likelihoods $P(D|S)$, $P(D|M)$, an $P(D|U)$[1]

(b) Find the three posterior probabilities $P(S|D)$, $P(M|D)$, and $P(U|D)$.

[1] You'll have to casually equate a probability to a frequency. That's often okay, but they aren't the same concept — they are two separate concepts whose values are being equated by assumption.

Question 3

In her book 'Complete Vocal Technique', Cathrine Sadolin divided the singing voice into four 'modes' which she called neutral, curbing, overdrive, and edge. These all have a different sound to them. Also, each mode only works on a certain set of vowel sounds. Specifically:

- Any vowel can be used in neutral.
- In curbing, only 'OO' (as in too), 'O' (as in book), and 'UH' (as in hungry) can be used.
- In overdrive, only 'EH' (as in egg) and 'OH' (as in so) can be used.
- In edge, only 'A' (as in cat), 'I' (as in sit), 'EH' (as in egg), and 'OE' (as in herb) can be used.

You observe the data D_1: a singer sang 'EH', and you want to know what the mode was. Use the following hypotheses and prior probabilities:

Hypothesis	Prior probability
N (Neutral)	0.5
C (Curbing)	0.1
O (Overdrive)	0.2
E (Edge)	0.2

(a) Calculate the posterior probabilities of N, C, O, and E given the data D_1. You may assume a likelihood of 0.1 for N, and that all legal vowels are equally likely for the other modes.

(b) Write down the marginal likelihood formula in terms of the specific statements used in this question.

The modes are also different in terms of the volume (how loud they are):

- Neutral can be quiet, medium, or loud.
- Curbing can only be medium.
- Edge can only be loud.
- Overdrive can only be loud.

Suppose the observed note was loud (call this statement D_2).

(c) Calculate the four posterior probabilities given D_1 and D_2. You may assume that for each mode, all allowed volumes are equally likely. Do the calculation sequentially, by taking the old posterior probabilities and using them as prior probabilities.

(d) Re-do (e) but do it by starting from the original prior and by updating using D_1 and D_2 simultaneously.