REFINING THE RELATIVISTIC MODEL FOR GAIA: COSMOLOGICAL EFFECTS IN THE BCRS

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ABSTRACT

This paper represents a first attempt of embedding the Barycentric Celestial Reference System, the fundamental relativistic reference system to be used for the modelling of Gaia observations, into some cosmological background. The general Robertson-Walker metric is transformed into local coordinates where the cosmological effects are represented as tidal potentials. A version of a cosmological BCRS is then suggested to lowest order. The effects of cosmological background on the motion of the Solar System are estimated and found to be completely negligible. The relation to the de Sitter and Schwarzschild-de Sitter solutions is discussed.

Key words: Relativity; Cosmology; Gaia reference frame.

1. INTRODUCTION

The relativistic model for Gaia published by Klioner (2003) is formulated in the first post-Newtonian approximation under the assumption that the Solar System is isolated. The relativistic reference system where the positions and other parameters of the sources are described is the Barycentric Celestial Reference System (BCRS) of the International Astronomical Union (Soffel et al. 2003). This model can be refined in two principal ways: (1) one can account for various sources of gravitational fields generated outside the Solar System (cosmology, microlensing, and gravitational waves); (2) one can optimize the post-Newtonian formulas to allow a faster numerical implementation of the model. Several projects aimed at such refinements are underway. Here we discuss the imbedding of the BCRS into some cosmological background.

2. THE BCRS AND EXTERNAL GRAVITATIONAL FIELDS

The BCRS is constructed under the assumption that the Solar System is isolated. One neglects herewith two kinds of effects:

(1) tidal forces due to any particular external body (e.g., nearby stars or the Galaxy),

(2) effects of cosmological background.

The tidal forces can be easily taken into account using the same theoretical framework as that used to construct the BCRS (e.g., Soffel et al. 2003). These effects were estimated to be negligible even for Gaia accuracy and, thus, the corresponding terms have been dropped in the BCRS metric tensor. To include effects from the cosmological background a new approach will be necessary.

3. THE ROBERTSON–WALKER METRIC IN THE VICINITY OF AN OBSERVER

Let us first consider a general Robertson-Walker metric in the neighbourhood of an observer, the size of the neighbourhood being about the size of the Solar System. Thus, assuming the usual conditions for homogeneity and isotropy the cosmological metric can be shown to coincide with the well-known Robertson–Walker metric which in spherical coordinates $(x^0 = c\,t, \rho, \theta, \varphi)$ reads

$$ds^2 = -c^2\,dt^2 + a^2(t) \times \left(\frac{d\rho^2}{1-k\,\rho^2} + \rho^2\,d\theta^2 + r^2\,\sin^2\theta\,d\varphi^2\right), \quad (1)$$

with $k = -1$, 0 or $+1$ being the curvature parameter. By a simple transformation of the radial coordinate

$$\rho = r\,\left(1 + \frac{1}{4}\,k\,r^2\right)^{-1} \quad (2)$$

this metric can be transformed into isotropic form

$$\begin{aligned} ds^2 &= g_{\alpha\beta}\,dx^\alpha\,dx^\beta, \\ g_{00} &= -1, \\ g_{0i} &= 0, \\ g_{ij} &= a^2(t)\,\left(1+\frac{1}{4}\,k\,r^2\right)^{-2}\,\delta_{ij}, \end{aligned} \quad (3)$$

where $(x^0 = c\,t, x^i)$, $r = |x^i|$ are Cartesian coordinates.

It is clear that the following coordinate transformations

$$T = t + \sum_{s=1}^{\infty} C_s(t) \left(\frac{a(t)\, r}{c}\right)^{2s}, \tag{4}$$

$$X^i = a(t)\, x^i \left(1 + \sum_{s=1}^{\infty} D_s(t) \left(\frac{a(t)\, r}{c}\right)^{2s}\right) \tag{5}$$

can be used to transform the metric (3) into a form (this form will be called *local* metric below) where the non-Minkowskian terms appear only as powers of radial distance from the origin

$$\begin{aligned} G_{00} &= -1 + \sum_{s=1}^{\infty} A_s(T) \left(\frac{R}{c}\right)^{2s}, \\ G_{0i} &= 0, \\ G_{ab} &= \delta_{ab} \left(1 + \sum_{s=1}^{\infty} B_s(T) \left(\frac{R}{c}\right)^{2s}\right), \end{aligned} \tag{6}$$

where $X^0 = c\,T$, X^i are the local coordinates, $R = |X^i|$, and A_s, B_s, C_s and D_s are unknown functions to be determined from the tensorial transformation law:

$$g_{\varepsilon\lambda}(t, \mathbf{x}) = \frac{\partial X^\mu}{\partial x^\varepsilon} \frac{\partial X^\nu}{\partial x^\lambda} G_{\mu\nu}(T, \mathbf{X}). \tag{7}$$

The matching procedure allows one to derive the unknown functions in a unique way. Let us introduce the following notations:

$$p_s = \frac{1}{a(t)} \frac{d^s}{dt^s}\, a(t), \tag{8}$$

$$f = \frac{k\, c^2}{a^2(t)}. \tag{9}$$

Then one has

$$\begin{aligned} A_1 &= p_2, \\ A_2 &= \frac{1}{4} \left(p_1^4 - 2\, p_1^2\, p_2 - p_2^2 + f\,(p_1^2 - p_2)\right), K \\ B_1 &= -\frac{1}{2} \left(p_1^2 + f\right), K \\ B_2 &= -\frac{1}{16} \left(p_1^4 - 4\, p_1^2\, p_2 - 2\, f\, p_1^2 - 3\, f^2\right), \\ C_1 &= \frac{1}{2}\, p_1, \\ C_2 &= \frac{1}{8}\, p_1\, (2\, p_2 - f), \\ D_1 &= \frac{1}{4}\, p_1^2, \\ D_2 &= \frac{1}{16}\, p_1^2\, (2\, p_2 - f), \\ &\dots . \end{aligned} \tag{10}$$

It is clear already from the dimensionality consideration that the coefficients A_s, B_s, C_s, D_s are homogeneous polynomials of the differentiation operator d/dt acting on $a(t)$ and f weighted as a term of degree two. The degrees of the polynomials A_s, B_s, C_s, D_s are $2s$, $2s$, $2s-1$ and $2s$.

Note that A_s and B_s are functions of T in (6). The above formulas (10) for these quantities should be understood in such a way that t should be formally replaced by T in the right-hand side of each equation, e.g., $A_1(T) = \left.\frac{\ddot a(t)}{a(t)}\right|_{t\to T}$. A program for *Mathematica* written by the authors allows one to perform the matching procedure to arbitrary order of r.

Thus, we have constructed the general Robertson-Walker metric in local coordinates where the cosmological effects manifest themselves only as tidal potentials.

4. A PARTICULAR CASE: THE FLAT DE SITTER SOLUTION

In the particular case of a flat de Sitter universe (flat empty universe with a Λ term) one has $a(t) = \exp(c\,\sqrt{\Lambda/3}\,t)$ and $k = 0$ in (3), so that the metric reads

$$\begin{aligned} g_{00} &= -1, \\ g_{0i} &= 0, \\ g_{ij} &= \delta_{ij}\, \exp\left(c\,\sqrt{\frac{2\Lambda}{3}}\, t\right). \end{aligned} \tag{11}$$

It is also well known that the flat de Sitter solution can be written in another form

$$\begin{aligned} ds^2 &= -A\, c^2\, dT^2 + A^{-1}\, d\rho^2 + \rho^2\, (d\theta^2 + \sin^2\theta d\varphi^2), \\ A &= 1 - \frac{1}{3}\, \Lambda\, \rho^2, \end{aligned} \tag{12}$$

which under the transformation

$$R = \frac{2\,\rho}{1 + \sqrt{1 - \frac{1}{3}\,\Lambda\,\rho^2}}, \tag{13}$$

$$\rho = \frac{R}{1 + \frac{1}{12}\,\Lambda\, R^2}, \tag{14}$$

can be re-written in isotropic form

$$\begin{aligned} ds^2 &= G_{\alpha\beta}\, dX^\alpha\, dX^\beta, \\ G_{00} &= -\left(\frac{1 - \frac{1}{12}\,\Lambda\, R^2}{1 + \frac{1}{12}\,\Lambda\, R^2}\right)^2, \\ G_{0i} &= 0, \\ G_{ab} &= \delta_{ab}\, \frac{1}{\left(1 + \frac{1}{12}\,\Lambda\, R^2\right)^2}. \end{aligned} \tag{15}$$

It is easy to see that (15) coincides with (6) in the case $a(t) = \exp\left(c\,\sqrt{\Lambda/3}\, t\right)$ and $k = 0$.

It is also known that the transformation (Lemaître 1925; Robertson 1928; Tolman 1934)

$$t = T + \frac{1}{2c}\sqrt{\frac{3}{\Lambda}}\, \log\left(1 - \frac{1}{3}\,\Lambda\,\rho^2\right), \tag{16}$$

$$r = \frac{\rho}{\sqrt{1 - \frac{1}{3}\,\Lambda\,\rho^2}}\, \exp\left(-\sqrt{\frac{\Lambda}{3}}\, c\,T\right) \tag{17}$$

can be used to transform (12) into (11) and vice versa. Inverting (16)–(17) one gets

$$T = t - \frac{1}{2c}\sqrt{\frac{3}{\Lambda}}\,\log\left(1 - \frac{1}{3}\,\Lambda\,a^2\,r^2\right), \qquad (18)$$

$$\rho = a\,r \qquad (19)$$

with $a(t) = \exp\left(c\,\sqrt{\Lambda/3}\,t\right)$. Combining (18)–(19) with (14) one gets the transformation between the two isotropic versions of the flat de Sitter metric (11) and (15):

$$T = t - \frac{1}{2c}\sqrt{\frac{3}{\Lambda}}\,\log\left(1 - \frac{1}{3}\,\Lambda\,a^2\,r^2\right), \qquad (20)$$

$$R = \frac{2\,a\,r}{1 + \sqrt{1 - \frac{1}{3}\,\Lambda\,a^2\,r^2}}. \qquad (21)$$

The coefficients of the series expansions of (20)–(21) coincides with the corresponding coefficients (10) in (4)–(5). This gives an additional check of the general formulas given in Section 3.

5. THE COSMOLOGICAL BCRS METRIC TO LOWEST ORDER

The cosmological BCRS metric can be considered as a perturbation of the corresponding cosmological metric. The gravitational field of the central body (or a system of bodies, e.g., of the Solar System) should be embedded into the cosmological metric. Neglecting the interaction of the cosmological fluid (including the cosmological constant) with the Solar System matter one gets the following simple version of the cosmological BCRS metric:

$$g_{00} \approx -1 + \frac{2}{c^2}\,w(t,\mathbf{x}) - \frac{2}{c^4}\,w^2(t,\mathbf{x}) + \frac{1}{c^2}\,A_1(t)\,|\mathbf{x}|^2,$$

$$g_{0i} \approx -\frac{4}{c^3}\,w^i(t,\mathbf{x}),$$

$$g_{ij} \approx \delta_{ij}\left(1 + \frac{2}{c^2}\,w'(t,\mathbf{x}) + \frac{1}{c^2}\,B_1(t)\,|\mathbf{x}|^2\right), \qquad (22)$$

where w and w^i are the usual BCRS metric potentials. Here we neglected: (1) higher post-Newtonian terms $\mathcal{O}(c^{-5})$ in g_{00} and in g_{0i}, and $\mathcal{O}(c^{-4})$ in g_{ij} due to post-post-Newtonian (and higher order) effects from the Solar System matter; (2) higher-order cosmological terms $\mathcal{O}(|\mathbf{x}|^4)$ proportional to A_s and B_s with $s \geq 2$, (3) any terms induced by the interaction between the Solar System matter and the cosmological fluid. For the case when the only source of energy-momentum tensor is the solar system matter and the field equations contain the cosmological constant Λ Soffel & Klioner (2003) have explicitly demonstrated how the metric (22) can be derived from the field equations in the corresponding approximation.

6. THE LOCAL EXPANSION HYPOTHESIS

The apparently simple question whether the cosmological expansion happens also locally (say, whether a hydrogen atom or the Solar System also expand) is a very complicated one and presents still an unsolved problem. Starting from Einstein himself different authors got different answers using different arguments (see Bonnor (2000) for a review of recent progress). Certainly, the answer to this question crucially depends on our model of the matter in the universe.

The cosmological BCRS metric suggested above implies that the cosmological expansion has certain influence on the properties of space-time within the Solar System. This is certainly true for the terms coming from the cosmological constant Λ or vacuum energy since this energy source is present everywhere. Note that recent cosmological observations suggest that about 73% of the energy in the universe comes from that source. The applicability of the suggested BCRS metric to the 4% coming from the luminous matter and the 23% of the dark matter has to be investigated further. Ideas of cosmological averaging might be employed here (Futamase 1996).

7. THE SCHWARZSCHILD–DE SITTER SOLUTION

The well-known Schwarzschild–de Sitter solution provides additional arguments in justifying the cosmological BCRS metric. The standard form of the Schwarzschild–de Sitter metric resembles (12):

$$ds^2 = -\mathcal{A}\,c^2\,dT^2 + \mathcal{A}^{-1}\,d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

$$\mathcal{A} = 1 - \frac{2m}{\rho} - \frac{1}{3}\,\Lambda\,\rho^2. \qquad (23)$$

This solution corresponds to the Schwarzschild solution with mass m for the field equations with cosmological constant. For $\Lambda = 0$ this solution coincides with the Schwarzschild solution in standard coordinates, and for $m = 0$ this agrees with the de Sitter solution (12). Robertson (1928) has shown that the metric (23) under the transformations (in this section $a(t) = \exp(c\,\sqrt{\Lambda/3}\,t)$)

$$T = t + \frac{8m^2}{c}\sqrt{\frac{\Lambda}{3}}\,F\left(\frac{1 - \frac{m}{2\,a(t)\,r}}{1 + \frac{m}{2\,a(t)\,r}}\right), \qquad (24)$$

$$F(x) = \int \frac{dx}{(1-x^2)\left(x^2\,(1-x^2)^2 - \frac{4}{3}\,m^2\,\Lambda\right)}, \qquad (25)$$

$$\rho = a(t)\,r\left(1 + \frac{m}{2a(t)\,r}\right)^2 \qquad (26)$$

can be put into form

$$ds^2 = g_{\alpha\beta}\,dx^\alpha\,dx^\beta,$$

$$g_{00} = -\left(\frac{1 - \frac{m}{2\,a(t)\,r}}{1 + \frac{m}{2\,a(t)\,r}}\right)^2,$$

$$g_{0i} = 0,$$
$$g_{ij} = \delta_{ij}\, a(t)^2 \left(1 + \frac{m}{2\, a(t)\, r}\right)^4. \qquad (27)$$

For $\Lambda = 0$ this metric coincides with the Schwarzschild solution in isotropic coordinates, and for $m = 0$ this metric gives the de Sitter solution in the form (11). It is important to note that Equations 24–26 coincide with (18)–(19) for $m = 0$.

Metric (23) written in isotropic or harmonic gauge represents an exact solution of the cosmological BCRS in local coordinates (cosmological terms as tidal potentials only) for the particular case of the de Sitter metric with a spherically symmetric perturbation. This solution can be used to verify the cosmological BCRS metric in higher approximations. The details will be published elsewhere.

8. DYNAMICAL COSMOLOGICAL EFFECTS IN THE SOLAR SYSTEM

The cosmological terms in the suggested BCRS metric lead to an additional central force described by the disturbing function

$$R = \frac{1}{2}\, A_1\, r^2. \qquad (28)$$

The parameter A_1 can be calculated as

$$A_1 = p_2 = \frac{\ddot{a}}{a} = -q\, H^2, \qquad (29)$$

where H and q are the Hubble constant and the deceleration parameter, respectively. Using the current best estimates of the parameters $q \approx -0.6$ and $H \approx 71$ km/s/Mpc one gets

$$A_1 \approx 3.2 \times 10^{-23}\ \mathrm{s}^{-2}. \qquad (30)$$

This is the value of A_1 at the present moment. The time dependence of this quantity is different for different cosmological models. In the flat de Sitter universe A_1 is time independent: $A_1 = \frac{1}{3}\, c^2\, \Lambda = \mathrm{const}$. In general A_1, however, is time dependent. At the distance of Pluto ($r = 40$ AU) the maximal value of the disturbing cosmological acceleration amounts to $A_1 \times 40$ AU $\approx 2 \times 10^{-23}$ m/s^2. Let us check how the cosmological perturbations under study change the orbit of planets. Since the disturbing function (28) gives a central force, the osculating inclination i, osculating longitude of the node Ω, and osculating semi-latus rectum $p = a(1 - e^2)$, a and e being the semi-major axis and the eccentricity, remain constant:

$$i = \mathrm{const}, \qquad (31)$$
$$\Omega = \mathrm{const}, \qquad (32)$$
$$p = a\,(1 - e^2) = \mathrm{const}. \qquad (33)$$

Since the time scale over which A_1 is changing is much larger than typical orbital periods in the Solar System, we might use an adiabatic approximation and put $A_1 = \mathrm{const}$ for the calculations of the other osculating elements. In this case one gets that the secular changes of averaged argument of perihelion $\overline{\omega}$ and averaged mean anomaly $\overline{M}_0$ read

$$\Delta\overline{\omega} = \frac{3\sqrt{1 - e^2}}{4\,\pi}\, A_1\, P^2\ \text{rad/revolution}, \qquad (34)$$
$$\Delta\overline{M}_0 = -\frac{7 + 3e^2}{4\,\pi}\, A_1\, P^2\ \text{rad/revolution}. \qquad (35)$$

Here and below P is the period of motion of the body under consideration. There are no secular effects in the semi-major axis and eccentricity and the amplitudes of periodic effects in these elements may amount to

$$\Delta_e = \frac{1 - e^2}{4\,\pi^2}\, A_1\, P^2, \qquad (36)$$
$$\Delta_a / a = \frac{2e}{1 - e^2}\, \Delta_e = \frac{e}{2\pi^2}\, A_1\, P^2. \qquad (37)$$

This shows that the secular perturbations in the argument of perihelion ω and in the mean anomaly M for Pluto are about $10^{-5}\mu$as per century and three orders of magnitude smaller for Mercury. The amplitudes of the periodic effects in a and e are about 10^{-17} for Pluto and 10^{-23} for Mercury. Therefore, these cosmological perturbations are completely negligible for the Solar System dynamics over times much less than the age of the Universe.

9. FURTHER WORK

Let us mention two almost independent problems which should still be solved:

(1) A better version of the cosmological BCRS metric should be constructed where the non-linearity of the field equations is taken into account.

(2) Parameters (e.g., parallaxes of distant sources) obtained while processing observations with the standard (non-cosmological) BCRS should be related to the parameters meaningful in cosmological context. Let us note here, that the parallax distance defined as $d_p \approx a(t)\, r \approx R$ agrees with the coordinate distance in our local coordinates.

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