GAIA OPTICAL ABERRATIONS DESCRIBED BY MEANS OF ORTHOGONAL POLYNOMIALS

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ABSTRACT

We present a method to describe the wavefront aberration distribution generated by a rectangular aperture with polynomial representations different from the usual Zernike one. While Zernike polynomials can still be used for wavefront decomposition over pupil shapes other than circular, in this case they form a basis that is no longer orthogonal. We compare the Zernike description with two new bases that are orthogonal over rectangular apertures, obtained with different approaches: the Gram-Schmidt orthogonalisation and the extension at two dimensions of the Legendre polynomials.

Key words: Gaia; Wavefront analysis; Orthogonal polynomials; Pupil apodisation.

1. INTRODUCTION

Gaia is the next generation global astrometric space mission. It will operate in continuous scanning mode to cover the full sky and to get a global reconstruction of the celestial sphere. General descriptions of the mission profile, hardware implementation and measurement principles can be found in other contribution to this meeting, (Perryman 2005; Mignard 2005; Pace 2005; Lindegren 2005). The current accepted configuration of the optical system, called Baseline Configuration, consists of two monolithic telescopes mounted on a common bench, each one looking at a different line of sight. Higher resolution is required along the scanning direction. The geometry of the aperture, for allocation purposes, is rectangular with the larger dimension (higher resolution) aligned with the scan direction. This layout gives an output wavefront that can still be described with the Zernike polynomials, but this choice is not optimal because they form a basis that is no longer orthogonal over such an aperture. In this paper, we compare the description of Gaia optical aberrations made with the use of three different sets of polynomials: standard circular Zernike polynomials, a set of polynomials obtained by Gram-Schmidt orthogonalisation over a rectangular pupil and a set derived from the classical Legendre polynomials.

2. GENERAL FRAMEWORK

To describe a wavefront using a complete set of functions Q_n, the criterion for the base selection imposes some constraints on the functions properties. One fundamental parameter is the root mean square (*rms*) of the wavefront, defined as:

$$
\begin{aligned}
rms = \sqrt{\langle (W - \overline{W})^2 \rangle} &= \sqrt{\langle W^2 \rangle - \overline{W}^2} \\
\langle W^2(\overline{x}) \rangle &= \frac{1}{A} \int W^2(\overline{x}) d\overline{x} \qquad (1) \\
W(\overline{x}) = \sum_n c_n Q_n(\overline{x}), \quad \|Q_n\|^2 &= \frac{1}{A} \int Q_n^2 d\overline{x}
\end{aligned}
$$

where A is the pupil area.

If the set Q_n is an orthogonal basis and $Q_1 = 1$, we have a simple formulation for the *rms*:

$$
\begin{aligned}
rms^2 = \frac{1}{A} \sum_{n,m} c_n c_m \int Q_n Q_m d\overline{x} &+ \\
- \left(\frac{1}{A} \sum_n c_n \int Q_n d\overline{x} \right)^2 &= \qquad (2) \\
\sum_n c_n^2 \|Q_n\|^2 - c_1^2 \|Q_1\|^2 &= \sum_{n>1} c_n^2 \|Q_n\|^2
\end{aligned}
$$

The Zernike polynomials Z_i form a complete orthogonal set over a circular pupil C, and $Z_1 = 1$. Therefore we can write

$$
\begin{aligned}
W(\rho, \theta) &= \sum_n c_n Z_n(\rho, \theta) \\
rms^2 &= \sum_{n>1} c_n^2 \|Z_n\|^2 \qquad (3) \\
C = \{(\rho, \theta) &\Rightarrow 0 \le \rho \le 1, 0 \le \theta \le 2\pi\}
\end{aligned}
$$

Each polynomial Z_i can be associated to a classical aberration, so it is easy to identify the contribution c_i of each term to the aberration function W. The piston corresponds to the Z_1 polynomial. On a rectangular pupil

$$R = \{(x, y) \Rightarrow -a \le x \le a, -b \le y \le b, a^2 + b^2 \le 1\}$$

obtained by apodisation of the original circular pupil (see Figure 1), the Zernike polynomials still describe the

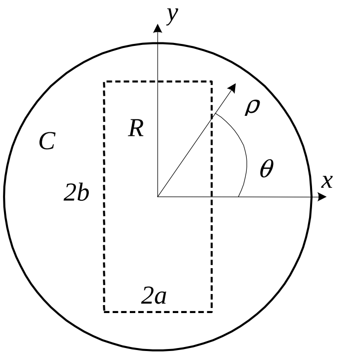

Figure 1. Pupil apodisation.

wavefront W, but they are no longer orthogonal. New sets of orthonormal functions can be derived from the Zernike description or built directly from 1-dimensional (1-D) polynomials (Courant & Hilbert 1953), as detailed here after, in order to satisfy Equation2.

Let P_i be an orthogonal basis over R, then

$$W = \sum_i c_i Z_i = \sum_j q_j P_j \tag{4}$$

or in matrix notation

$$W \approx c^T Z = q^T P \tag{5}$$

where, for practical reasons, the sum is made up to a finite number of polynomials N. The P_j basis can be expressed as well in terms of Zernike polynomials as

$$Z = BP \tag{6}$$

Introducing the matrix S, whose elements are given by the scalar product of the Zernike polynomials over the apodised pupil

$$S_{nm} = \int_R Z_n Z_m dxdy, \tag{7}$$

the B matrix is given by the equation

$$S = BDB^T \tag{8}$$

where D is the diagonal matrix defined by the normalisation condition of the P_j set. The coefficients q_j of the new wavefront decomposition satisfy the equation

$$q^T = c^T B \tag{9}$$

3. GRAM-SCHMIDT

A solution of the equation $S = BDB^T$ can be found by Gram-Schmidt orthonormalisation with $D = I$. Switching back to index notation, the B matrix can be built numerically by recursive application starting from the first diagonal element B_{11}. The elements of the S matrix are obtained from

$$S_{nm} = \int_{-a}^{a} \int_{-b}^{b} Z_n(x,y) Z_m(x,y) dxdy \tag{10}$$

Figure 2 shows the wavefront maps corresponding to each single term of the sum $W = \sum_j q_j^{gs} P_j^{gs}$ with $q_j^{gs} = 1$ for $j = 2, .., 21$.

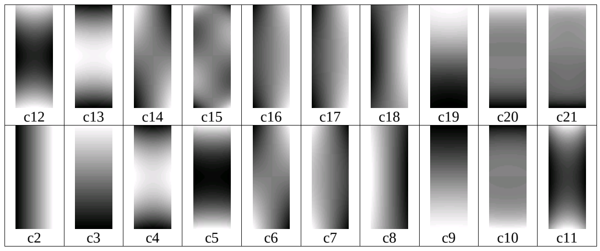

Figure 2. Wavefront maps for the Gram-Schmidt Polynomials P_j^{gs}, $j = 2, .., 21$. $j = 1$ (piston) is not shown.

4. 2-D LEGENDRE

A set of orthogonal polynomials can also be defined without the need for explicit calculation of the matrix S, as a linear combination of the classical 1-D Legendre polynomials:

$$L_{ij}(x,y) = L_i(x/a) L_j(y/b) \tag{11}$$

This set forms an orthogonal basis over the rectangular pupil, with norm

$$\|L_{ij}\|^2 = \frac{4ab}{(2i+1)(2j+1)} \tag{12}$$

In this case the matrix B can be obtained by direct analytical calculation. B is the product of two matrices defining the relationship between the Zernike, the Binomial and the 2-D Legendre bases. For a Zernike representation in polar vs. cartesian coordinates see for example Malacara (1978). As in the previous paragraph, Figure 3 shows the wavefront maps corresponding to each single term of the sum $W = \sum_{ij} q_{ij}^L L_{ij}$ with $q_{ij}^L = 1$ for $i > 0, j > 0$.

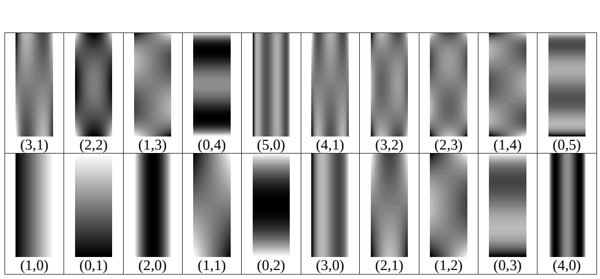

Figure 3. Wavefront maps for the 2D-Legendre Polynomials L_{ij}, $i > 0, j > 0$. $i = j = 0$ (piston) is not shown.

5. RESULTS

As an application of the previous concepts, we performed numerical calculations assuming a rectangular pupil R with apodisation parameters $a = 0.2$ and $b = 0.56$. This corresponds, for the Gaia case, to rectangular aperture sizes of 500 mm and 1400 mm respectively, scaled to an overall circular pupil C of diameter $2\rho = 2500$ mm, that is used for computations with the Zernike polynomials.

We consider three different wavefronts W:

- Case A: the wavefront W_A is built with all the Zernike coefficients taken equal;
- Case B: W_B is a realistic case of a singlet, the object being an on-axis point in the field of view;
- Case C: same optical design as case B, the object being an off-axis point in the field of view with angle coordinates $0.2°$ in the x direction and $0.3°$ in the y direction.

The values of the Zernike coefficients c_i for the three cases considered are shown in Table 1. Units for the coefficients (column three to five) are waves. The last two rows summarise the *rms* of the polynomials, calculated using Equation 3, and the 'real' wavefront *rms*, obtained by direct numerical evaluation. The value calculated using Equation 3 gives the wavefront *rms* as if evaluated over the original circular pupil rather than over the rectangular one. The discrepancy is big and dependent from the case considered. Such an effect has been emphasised on purpose by the choice of a circular pupil quite larger than the encircled rectangular one.

Table 2 and Table 3 show the expansion coefficients of the same wavefronts obtained using Equation 9, respectively for the 2-D Legendre and the Gram-Schmidt polynomials, and the *rms* calculated using Equation 2 compared again with the wavefront *rms* obtained by independent calculation. As it can be seen, the values are quite in good agreement, as expected.

It is also interesting to see how the total aberration of the wavefront is distributed over the single terms. This is shown in Figures 4, 5 and 6 respectively for the three considered cases. Each plot compares the Gram-Schmidt representation (diamonds) with the 2-D Legendre one (triangles).

In general the values cannot be compared one-by-one, because the individual polynomial of one basis can not be directly associated to the polynomial with the same identifier of the other basis. This is not true for the first three polynomials, that are in fact the same for both bases. Going to higher orders, the one-to-one correlation fails. Nevertheless, there is still a common feature, i.e., the *rms* evaluated using the coefficients of the polynomials up to a certain order (in other words the two different representations cannot transfer aberration weight from one order to another). This value, compared with the 'real' wavefront *rms*, also gives an estimate of how good the polynomial fit is. In the three considered cases, 21 polynomials give a good approximation of the wavefront, but for more complicated optical designs (as Gaia is) they may not be sufficient.

Table 1. Value of the Zernike coefficient c_i for the three considered cases (see text).

Order	i	W_A	W_B	W_C
0	1	0.1	0.9638	1.0534
1	2	0.1	0.0000	−0.7533
	3	0.1	0.0000	−1.1300
2	4	0.1	0.0000	−0.0689
	5	0.1	1.4455	1.5351
	6	0.1	0.0000	0.1655
3	7	0.1	0.0000	0.0000
	8	0.1	0.0000	−0.3765
	9	0.1	0.0000	−0.5648
	10	0.1	0.0000	0.0000
4	11	0.1	0.0000	0.0000
	12	0.1	0.0000	0.0000
	13	0.1	0.4817	0.4816
	14	0.1	0.0000	−0.0001
	15	0.1	0.0000	0.0000
5	16	0.1	0.0000	0.0000
	17	0.1	0.0000	0.0000
	18	0.1	0.0000	0.0001
	19	0.1	0.0000	0.0002
	20	0.1	0.0000	0.0000
	21	0.1	0.0000	0.0000
rms of poly		0.1663	0.8619	1.1645
wavefront *rms*		0.0415	0.0843	0.1628

Table 2. Value of the 2-D Legendre coefficient q^L_{ij} for the three considered cases (see text).

Order	(i,j)	W_A	W_B	W_C
0	(0;0)	0.0782	0.0659	0.0933
1	(1;0)	0.0345	0.0000	−0.0291
	(0;1)	0.0529	0.0000	−0.1913
2	(2;0)	−0.0133	0.0188	0.0217
	(1;1)	−0.0331	0.0000	0.0371
	(0;2)	−0.0233	0.1787	0.2307
3	(3;0)	−0.0036	0.0000	−0.0036
	(2;1)	−0.0197	0.0000	−0.0253
	(1;2)	0.0000	0.0000	−0.0473
	(0;3)	−0.0256	0.0000	−0.1191
4	(4;0)	0.0004	0.0011	0.0011
	(3;1)	0.0022	0.0000	0.0000
	(2;2)	0.0034	0.0322	0.0322
	(1;3)	0.0056	0.0000	0.0000
	(0;4)	0.0067	0.0650	0.0650
5	(5;0)	0.0001	0.0000	0.0000
	(4;1)	0.0006	0.0000	0.0000
	(3;2)	0.0000	0.0000	0.0000
	(2;3)	0.0038	0.0000	0.0000
	(1;4)	0.0000	0.0000	0.0000
	(0;5)	0.0042	0.0000	0.0000
rms of poly		0.0415	0.0835	0.1616
wavefront *rms*		0.0415	0.0843	0.1628

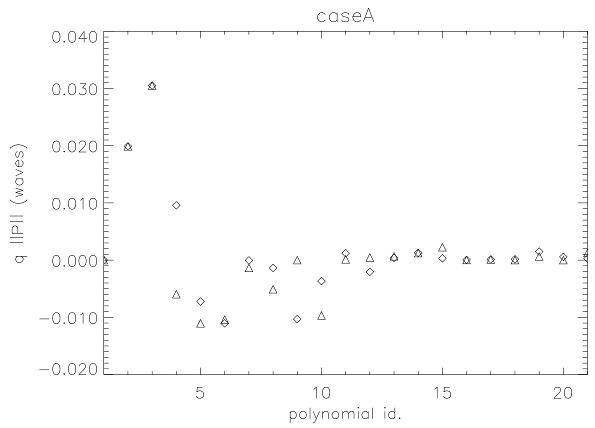

Figure 4. Wavefront W_A: comparison of the normalised magnitude of each polynomials coefficient for 2-D Legendre (triangles) and Gram-Schmidt (diamonds).

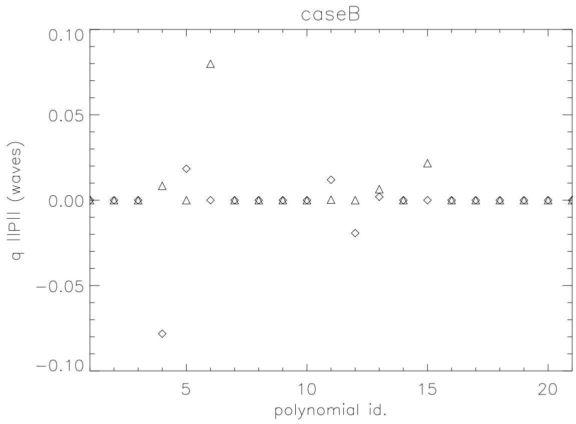

Figure 5. Wavefront W_B: comparison of the normalised magnitude of each polynomials coefficient for 2-D Legendre (triangles) and Gram-Schmidt (diamonds).

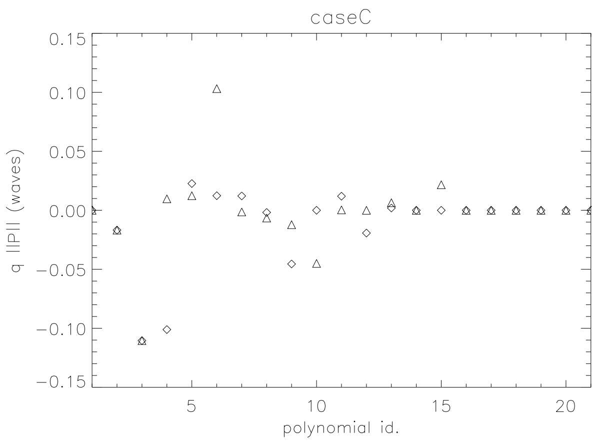

Figure 6. Wavefront W_C: comparison of the normalised magnitude of each polynomials coefficient for 2-D Legendre (triangles) and Gram-Schmidt (diamonds).

Table 3. Value of the Gram-Schmidt coefficient q_j^{gs} for the three considered cases (see text).

Order	j	W_A	W_B	W_C
0	1	0.0523	0.0441	0.0624
1	2	0.0133	0.0000	−0.0112
	3	0.0204	0.0000	−0.0739
2	4	0.0064	−0.0524	−0.0677
	5	−0.0048	0.0123	0.0152
	6	−0.0074	0.0000	0.0083
3	7	−0.0000	0.0000	0.0081
	8	−0.0009	0.0000	−0.0012
	9	−0.0069	0.0000	−0.0305
	10	−0.0025	0.0000	0.0000
4	11	0.0008	0.0080	0.0080
	12	−0.0014	−0.0129	−0.0129
	13	0.0003	0.0013	0.0013
	14	0.0009	0.0000	−0.0000
	15	0.0002	0.0000	0.0000
5	16	0.0000	0.0000	0.0000
	17	0.0001	0.0000	−0.0000
	18	0.0001	0.0000	0.0000
	19	0.0010	0.0000	0.0000
	20	0.0004	0.0000	0.0000
	21	0.0002	0.0000	0.0000
rms of poly		0.0415	0.0835	0.1616
wavefront *rms*		0.0415	0.0843	0.1628

CONCLUSIONS

We present a general method to describe the wavefront aberration distribution over a rectangular aperture, which is the case of Gaia, by means of orthogonal polynomials. The main advantage of such a representation is that the wavefront error can be easily derived from the coefficients of the series expansion. We discuss a method to obtain two new bases that are orthogonal over rectangular apertures: one derived from the classical 1-dimensional Legendre polynomials, and another from Gram-Schmidt orthogonalisation. As a practical application, we compare the outputs of the new wavefront descriptions with those of the standard Zernike decomposition, in three different cases.

We have now a very useful tool for description of optical aberrations over rectangular shaped apertures. This is crucial to have the correct optical analysis for the Gaia Baseline Configuration telescopes.

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