# CORRECTING SYSTEMATIC ERRORS IN THE DETERMINATION OF PROPER MOTIONS IN THE GALAXY

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## ABSTRACT

After a reminder of what is the aberration in proper motions as seen from the barycentre of the Solar System, different ways of correcting the observed proper motions are discussed using a simple model of galactic potential. A simulation shows that adding the information provided by radial velocities may greatly improve the determination of the correction and may also be used to improve the distances provided by the parallaxes. However, to do this, it necessary to have a model of the gravitational potential in the Galaxy.

Key words: Aberration; Proper motions; Parallaxes.

## 1. INTRODUCTION

In his presentation of this Symposium, François Mignard indicated that, at $\mu$as accuracies, one must reassess the classical definitions of parallaxes and proper motions. In the case of parallaxes, in addition to the necessity of decoupling their observations from the effect of light bending, it can be shown that the most probable distance is not $1/\varpi$, and that one must take the Lutz & Kelker approach (1973) as shown by Oudmaijer et al. (1998) and Kovalevsky (1998).

In the case of proper motions, the present situation is that the celestial reference frames (ICRF and its extensions such as HCRF) are barycentric, but defined with respect to a fixed ensemble of extragalactic objects. This means that the motion of the barycentre of the Solar System is assumed to be linear, which is not the case, since it revolves around the centre of the Galaxy in some 240 million years. This non-linearity of the motion is true for all stars of the Galaxy. In this article, we consider the consequences of these non-linearities and how their effects on instantaneous proper motions can be accounted for.

## 2. ABERRATION OF A STAR IN THE GALAXY

It is well known that the linear motion of the barycentre is of the order of 220 km s$^{-1}$, which produces an aberrational displacement of the position of stars of about $150''$. However, this displacement has no consequence on studies of kinematical and dynamical studies within the Galaxy and this fixed aberration is rightfully ignored.

Now, if we consider that this motion is not linear, the velocity vector rotates with time, and, consequently, the barycentric position of the observed object changes with time. The theory can be found in Kovalevsky (2003). Here is a short account.

Let us first assume that the motion of the star is circular at a distance $R$ of the centre of the Galaxy with an angular velocity

$$\omega = V_c/R \tag{1}$$

Let $S_0$ be the position of $S$ at time $t = t_0$, and set the axes of coordinates $S_0x$ (unit vector, $\mathbf{u}$), along $S_0G$, and $S_0y$ (unit vector, $\mathbf{v}$), perpendicular and in such a way that the rotation is direct (Figure 1).

At time $t_0$, the velocity $\mathbf{V_c}$ is parallel to $Gy$. At time $t$ in years, it has turned by $\omega t$, so that the components of the velocity vector $\mathbf{V}(t)$ are

$$\begin{aligned} V_x(t) &= V_c \sin\omega t = V_c\omega t + ... \\ V_y(t) &= V_c \cos\omega t = V_c - \frac{1}{2}\omega^2 t^2 + ... \end{aligned} \tag{2}$$

This produces an apparent displacement of $S$, $\Delta\mathbf{S}$ equal to

$$\Delta\mathbf{S} = -\mathbf{V}(t)/c \tag{3}$$

where $c$ is the speed of light. Each component in Equation 2 produces a component of the aberration:

1. $\Delta\mathbf{S}_0 = (-V_c/c)\mathbf{v}$ is the constant galactic aberration, in the $Gy$ direction, mentioned above.
2. $\Delta\mathbf{S}_1 = (-V_c\omega t/c)\mathbf{u}$ is along the direction $\mathbf{GS}$. It has the form of a proper motion. The yearly aberration in proper motion is, therefore:

$$\Delta\vec{\mu} = (-\frac{V_c\omega}{c})\mathbf{u} \tag{4}$$

3. $\Delta\mathbf{S}_2 = -V_c\omega^2t^2/2c\mathbf{v}$ is an additional displacement along $\mathbf{V}_0$. Being proportional to the square of $\omega$, it is negligible (for the Sun its value is 0.00022 $\mu$as yr $^{-1}$) and will not be considered here.

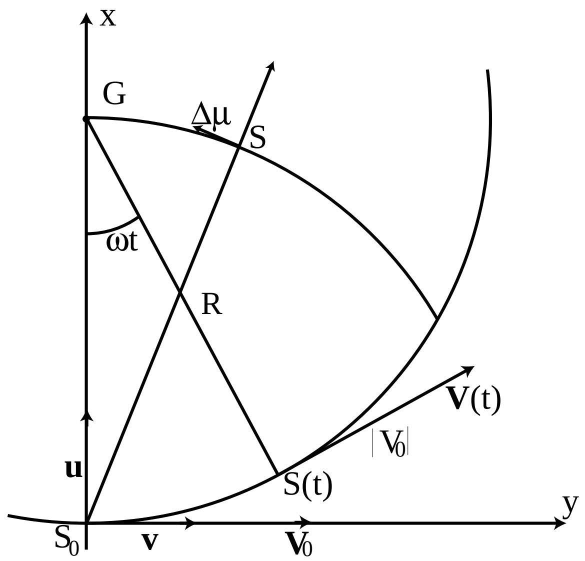


*Figure 1. Evolution of the velocity vector* $\mathbf{V}$.

In the two body problem, one has

$$-\omega V_c \mathbf{u} = -\frac{V_c^2}{R}\mathbf{u} = \frac{d\mathbf{V}}{dt} = \frac{d^2\mathbf{GS}}{dt^2} \qquad (5)$$

As a consequence, the equations are true also for non-circular orbits, the driver being the central acceleration. More so, any orbit, in a small environment can be represented by an osculating ellipse centred on a point in the direction of the force. Therefore, to determine the aberration in proper motion, at any point, it is sufficient to know the local force, that is the local galactic potential.

Note that, if one would have only a few hours of observation from the Earth, one would apply these equations for the planetary aberration.

## 3. APPLICATION TO THE ABERRATION IN PROPER MOTION DUE TO THE MOTION OF THE BARYCENTRE

We shall not repeat the results obtained in Kovalevsky (2003). Let us simply state that all stars and galaxies are affected by the aberration in proper motion by a quantity $\Delta\vec{\mu}$. Assuming $R = 8500$ pc, and a circular velocity $V_0 = 220$ km s$^{-1}$, one obtains, along $\mathbf{u}$,

$$\Delta\vec{\mu} = \tfrac{\Delta V_0}{c}\mathbf{u} = \sigma_0\mathbf{u} = 4.0\ \mu\text{as}\ \mathbf{u}\ \text{per year}$$

The aberration in the direction of a star $S$, as seen from the barycentre $B$ of the Solar System is the projection of this vector on the sky in the direction of $S$. If $\zeta$ is the angle $(\mathbf{BG}, \mathbf{BS})$ ( $(\mathbf{S_0G}, \mathbf{S_0S}$ in Figure 1), its value is

$$\Delta\mu = \frac{\Delta V_0}{c}\sin\zeta \qquad (6)$$

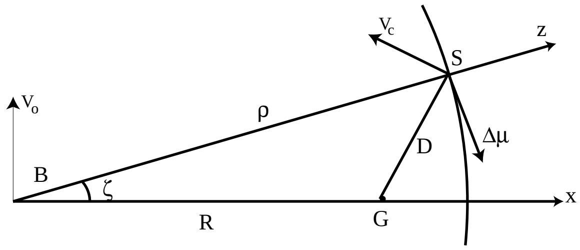


*Figure 2. Situation of the star* $S$ *in the Galaxy and the correction* $\Delta\mu_s$

In galactic coordinates, this gives

$$\begin{aligned} \mu_\lambda \cos\phi &= -\sigma_0 \sin\lambda\ \mu\text{as per year} \\ \mu_\phi &= -\sigma_0 \sin\phi\cos\lambda\ \mu\text{as per year} \end{aligned} \qquad (7)$$

This effect may be applied to galaxies, which present this apparent motion (ESA 2000).

## 4. ABERRATION IN PROPER MOTION OF STARS IN THE GALAXY

Until now, we have considered only the time-dependent aberration produced by the motion of the Sun in the Galaxy. In particular, the results given in the previous section are to be applied to the construction of the celestial reference frame based upon the observations of extragalactic objects, since they present such an apparent motion.

Now, if we consider a star $S$ which revolves around the centre of the Galaxy. The results of Sections 2 and 3 apply also to them and, in particular, Equations 3 and 4. The correction $\Delta\mu_s$ to the proper motion of a star $S$ at a distance $D$ from the galactic centre $G$ is along the great circle of the celestial sphere comprising $G$ and $S$ and centred at the barycentre $B$ of the Solar System (see Figure 2). Let $\rho$ be the distance $BS$ and $R = BG$. Let us also call $V_0$ and $V_c$ respectively the circular velocities of $B$ and $S$ and $c$ the speed of light, the total correction is obtained by subtracting the stellar part from the barycentric part.

$$\Delta\mu_s = k\left(\frac{{V_c}^2 R\sin\zeta}{cD^2} - \frac{{V_0}^2}{cR}\right) \qquad (8)$$

where $\zeta$ is the angle $\mathbf{BS}, \mathbf{BG}$, and $k$ is a numerical factor depending upon the units used. If the velocities are given in km s$^{-1}$ and the distances in pc, $k = 210\,900$.

The distance $D$ of the star to the centre of the Galaxy being squared in the denominator, the correction is quickly increasing if approaching $G$ and may amount to several times the uncertainty of the Gaia determinations of proper motions. This is exhibited by Figure 3. Conversely, the correction is very small in large regions around the Sun, so that, for instance at distances smaller than 2 kpc, it does not exceed 2 $\mu$as (see Kovalevsky 2003) and can generally be neglected.

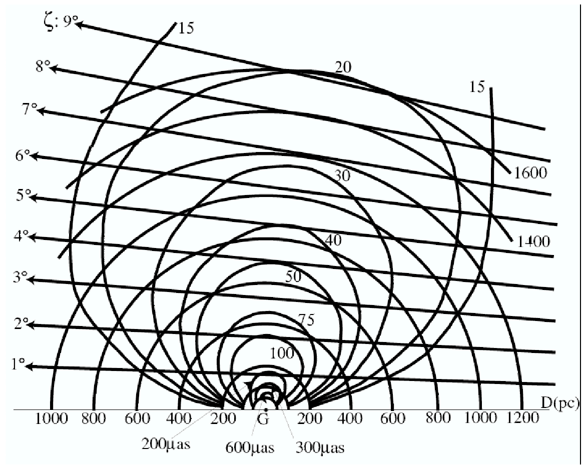


*Figure 3. Magnitude of the yearly aberration in proper motions due to the motion of the barycentre around the centre of the Galaxy. Arrows give the angular distance $\zeta$ from the direction of the centre of the Galaxy as seen from the Sun*

## 5. UNCERTAINTY BUDGET

**Remark**:The second term of Equation 8, due to the motion of the barycentre is close to 4 $\mu$as per year. Even if one admits an error of 10%, it remains negligible, so it is sufficient to concentrate on the first term. In addition, $k$, $c$, and $\zeta$ are well known parameters, while $R$ and $V_0$ are probably known to better than 5%. So, by far, the major part of the uncertainty arises from the uncertainty on $D$ and $V_c$. The latter depends on the potential of the Galaxy and is a function of $D$. So we set $V_c = V(D) = V$.

The choice of the potential model will be a crucial one. One may expect that a first reduction of Gaia data will help to improve it. But, even at the very first stage, one shall need some credible model of the galactic potential. The present models give a rather constant value for $V$ of the order of 200–220 km s$^{-1}$ between 1000 and 12 000 pc from $G$ (probable error 5%); one can refer among others, to Fich & Tremaine (1991) or Dehnen & Binney (1998).

It becomes more and more uncertain as one approaches $G$ and is practically unknown closer than 500 pc. Therefore, most of the uncertainty comes from the fact that $D$ is not known, except by its relation to $R$, $\zeta$, and $\rho$, the latter being in principle determined from the parallax $\varpi$ obtained from Gaia observations:

$$D^2 = R^2 + \rho^2 - 2\rho R \cos\zeta \tag{9}$$

Under these conditions,

$$\Delta\mu = \frac{k\sin\zeta}{c}\frac{V_c^2 R}{D^2} = \frac{k\sin\zeta}{c}A$$

and, differentiating, one gets

$$\begin{aligned}\delta(\Delta A) &= \frac{k}{D^4}\left[2RVD^2\delta V + V^2(\rho^2 - R^2)\delta R\right] \\ &+ \frac{2kRV^2}{D^4}(R\cos\zeta - \rho)\delta\rho \end{aligned} \tag{10}$$

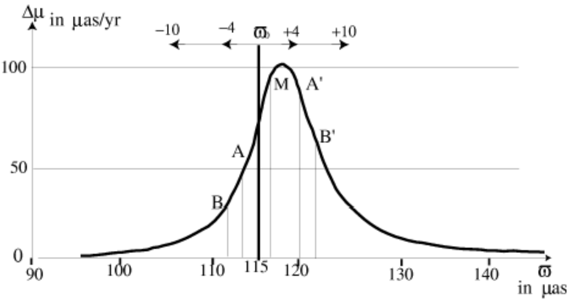


*Figure 4. Variation of $\Delta\mu$ as a function of the parallax in $\mu$as for $\zeta$ = 2° and intervals of confidence for an observed parallax of 115 $\mu$as with errors of 4 and 10 $\mu$as using the numerical method ($A - A'$ for $\delta\varpi$ =4 $\mu$as, $B - B'$ for 10 $\mu$as).*

Now, $V$ is given by the model $V = V(D)$. Calling $V' = \mathrm{d}V/\mathrm{d}D$ which, incidentally, also depends upon $R$ and $\rho$, one gets

$$\delta V = \frac{V'}{D}\left[(R\cos\zeta - \rho)\delta\rho + (\rho\cos\zeta - R)\delta R\right]$$

Finally, one obtains the following expression for the uncertainty of the correction:

$$\begin{aligned}\delta(\Delta\mu) = &\quad \tfrac{kV\sin\zeta}{cD^4}(V(\rho^2 - R^2)\delta R \\ + &\quad \tfrac{2kRVV'\sin\zeta}{cD^3}(\rho\cos\zeta - R))\delta R \\ + &\quad \tfrac{2RkV\sin\zeta}{cD^4}(R\cos\zeta - \rho)(V + DV')\delta\rho \end{aligned} \tag{11}$$

Since $R$ is relatively well known, the major part of the error budget on the determination of $\Delta\mu$ is the second part which depends upon

$$\delta\rho = -\delta\varpi/\varpi^2 = -\rho^2\delta\varpi$$

In the vicinity of the centre of the Galaxy, the parallax is of the order of 100–120 $\mu$as. If there is a 10 $\mu$as error in the parallax, this represents an error of 1000 pc in $\rho$. So, a linearized representation of the errors in determining $\Delta\mu$ is totally incorrect.

One can use a numerical approach to evaluate these errors by computing the values of $\Delta\mu$ in the interval $|\varpi_0 + \delta\varpi|$ and $|\varpi_0 - \delta\varpi|$ and convoluting it with the probability distribution function (pdf) of $\varpi$. This provides an estimation of the error on the computation of uncertainties on $\Delta\mu$ expressed in $\mu$as per year as functions of $\varpi_0$ and $\zeta$ around the centre of the Galaxy. An illustrative example is given in Figure 4 drawn with $\zeta = 2°$. It was assumed that the observation gave $\varpi_0 = 115$ $\mu$as. The one sigma uncertainty domains that were obtained are drawn for $\delta\varpi = 4$ $\mu$as and 10 $\mu$as.

The interval of confidence given in this Figure is much smaller than the estimation for $\delta\Delta\mu$ obtained by applying directly $\delta\varpi = 4$ $\mu$as in Equation 7.

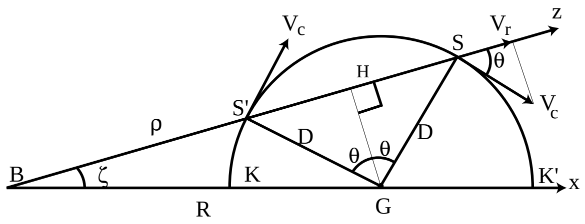


*Figure 5. Radial and circular velocities at a distance $D$ of $G$ in a given direction*

## 6. USE OF RADIAL VELOCITIES

The values of the correction of aberration in proper motion, are in some cases, much larger than the observation uncertainties. A possible way to improve the situation is to use another piece of information provided by Gaia, namely the proper motions.

Let us first assume that the orbit of the star is circular centred at $G$. Let us call $\theta$ the angle the velocity and the direction of the star. At a distance $D$ from $G$, there are two possible positions $S$ and $S'$ having the same circular velocity $V_c$ (see Figure 5). The radial velocity is also the same and equal to $V_r = V_c \cos\theta$. The direction of the star is known and defined by $\zeta$. If $H$ is the projection of $G$ on $Bz$, one has $GH = R\sin\zeta$ and

$$HS = GH \tan\theta = R \sin\zeta \tan\theta$$

It results that the distances $\rho$ and $\rho'$ of $S$ and $S'$ are given by

$$\rho = R\cos\zeta \pm R\sin\zeta\sqrt{\left(\frac{V_c}{V_r}\right)^2 - 1} \qquad (12)$$

The first term being known with sufficient accuracy, after some algebra, the error on $\rho$ is given by

$$\delta\rho = \pm\left(\frac{R\sin\zeta V_c}{V_r^3\sqrt{V_c^2 - V_r^2}}\right)(V_r\delta V_c - V_c\delta V_r) \qquad (13)$$

If we assume that $V_c$ does not depend significantly from a constant value, then, $\delta\rho$ is given by

$$\delta\rho = \pm\left(\frac{R\sin\zeta V_c^2}{V_r^3\sqrt{V_c^2 - V_r^2}}\right)\Delta V_r \qquad (14)$$

which can be written as

$$\delta\rho = \frac{RV_c\sin\zeta}{V_r\sin\theta}\frac{\delta V_r}{V_r}$$

Returning to $\Delta\mu$, one gets, after some computation and noting that $R\cos\zeta - \rho = SH = D\sin\theta$, and $R\sin\zeta = D\cos\theta$, the folowing expression

$$\delta(\Delta\mu) = \frac{2kV_c^2 R\sin\zeta}{cD^2}\frac{\delta V_r}{V_r} \qquad (15)$$

This result, compared with the first term of Equation 8, shows that the error on $\Delta\mu$ is its value multiplied by $2\delta V_r/V_r$:

$$\delta(\Delta\mu) = \frac{2\delta V_r}{V_r}\Delta\mu \qquad (16)$$

The gain in precision in the determination of $\Delta\mu$ depends directly on the relative precision of the determination of the radial velocity of the star. Radial velocities in the central regions of the Galaxy are expected to be mostly in the range of 100–200 km s$^{-1}$, with uncertainties of the order of 10–15 km s$^{-1}$ up to limiting magnitude 17. The factor $2\delta V_r/V_r$ will be smaller than 0.2 so that the uncertainty on $\Delta\mu$ will be a small part of the correction to be applied. This situation is definitely not true if one uses directly the distance derived from the parallax.

To use radial velocities, one will compute $\rho$ from Equation 12 that will give two possible distances, but only one $D$ using Equation 9 (in the circular motion assumption). Then, $V_c$ will be derived from the expression of the potential at distance $D$ from $G$. The main cause of error is due to the lack of knowledge of $V$. However, the uncertainties obtained by this method appear to be smaller than in the methods using parallaxes only. In any case, both should be used and possibly combined.

As an example, assuming $\zeta = 2°$; $D = 500$ pc and $V_c = 200\pm$ 10 km s$^{-1}$, one gets $\Delta\mu = 34\pm4$ $\mu$as.

**Remark**: It is interesting to note that, comparing the two distances $\rho$ derived from Equation 12 with the value derived from the parallax, one may reject one of them. The remaining $\rho$, which has a much smaller uncertainty, can be used to improve the parallax given by astrometry.

## 7. CONCLUSION

Many questions should be answered before one can safely determine the galactocentric proper motions in the central parts of the Galaxy. Let us mention some.

1. Is there, or will there be a good model of galactic absorption around the centre of the Galaxy, so that one could infer from photometry and spectroscopy the type (and therefore the absolute magnitude) of stars providing parallaxes?

2. In the determination of circular velocities by radial velocities an additional error comes from the radial dispersion of velocities. Are there some hints about the velocity ellipsoids for particular types of stars in the central regions of the Galaxy? Would the centre of the ellipsoid correspond to the circular velocity?

3. One must derive the error model to a high order of the parameters so as to replace, if possible, the numerical approach described in Section 5.

4. One must generalize the results in order to take into account a more sophisticated potential model, in particular out of the galactic plane and in the bar.

A simulation of these situations is being considered and prepared.

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