DIRECT FRINGE DETECTION AND SAMPLING OF THE DIFFRACTION PATTERN

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ABSTRACT

The Fizeau interferometer proposed in the GAIA concept adds fine structures or fringes to the diffraction pattern of a single aperture, thus improving the precision in the position measurement of the drifting pattern. As direct fringe detection is by far most favoured, we investigate the precision in terms of baseline and sampling frequency, the noise source being photon noise only. The detector, supposed to be a CCD camera, operates in the time delayed integration mode. Considering only the filtering effect due to the finite width of a pixel, an optimum baseline is obtained with two samples per fringe period. In such a case the undersampling can be corrected exactly if two neighbouring cells are driven with a clock offset of half a sample. Two linear combinations of these measurements will give an unbiased estimate for the position of the diffraction pattern, without any loss of information.

Key words: space astrometry; GAIA, Fizeau interferometer; CCD detectors

1. INTRODUCTION

The GAIA concept is based on three telescopes observing at wide angles. Each of the telescopes has two distinct entrance pupils and operates as a Fizeau stellar interferometer. Due to the interference fringes in the image plane, each interferometer is expected to have high precision in the position measurements of star images along the scanning direction. The field of view of such an interferometer being as large as the aberration-free field of view of the telescope, the Fizeau interferometer appears to be ideally suited for mapping the sky from a spinning spacecraft.

As for any astrometric measurement, high stability and accuracy of the telescope and detection system are required. Direct fringe detection in the focal plane would be favoured due to its greater simplicity and efficiency, but seemed to be unrealistic with the parameters choosen in the GAIA baseline definition study (Lindegren & Perryman 1994). Indeed, the image scale in the focal plane and the fringe period must 'fit' the pixel width in the scanning direction.

The purpose of the detector is to record the necessary information for accurate position measurements, that is without bias, and with the largest Signal/Noise ratio. The noise source will be photon noise only, neglecting read-out noise. What *fit* will mean for a CCD detector operating in the time delayed integration mode is the subject of this paper.

In a first section we evaluate the precision in terms of baseline for an ideal detector, that is with an infinitely small width. In a second section, we evaluate the optimum baseline or fringe frequency for a finite pixel width. At this optimum frequency, the fringe pattern is undersampled and we show how to recover its position without bias, and without loss in the signal-to-noise ratio.

2. FIZEAU STELLAR INTERFEROMETER WITH AN IDEAL DETECTOR

The maximum precision that can be achieved in the location of a star image is evaluated with the help of the optimum weighting function (Lindegren 1978). The problem is a one dimensional problem, along the path of the image, and an ideal photon counting detector is supposed to be used in the focal plane of the Fizeau interferometer.

Let v be the angular coordinate along this path in the instrument frame. At any time t, the observed interferogram is $g_0(v - v_0)$ with $v_0 = \omega_0 t$, ω_0 being the scanning speed of the spacecraft.

Let $u = v - v_0$ the angular coordinate along the baseline projection in the moving stellar frame. The optimum weighting function for the location of a pattern with photon noise only is:

$$W(u) = \frac{g_0'(u)}{g_0(u)}$$

and the precision in the location, taken as the inverse of the variance is:

$$\frac{1}{\sigma_p^2} = \int \mathrm{d}u \frac{[g_0'(u)]^2}{g_0(u)} \qquad (1)$$

The expression for the interferogram $g_0(u)$ with two identical apertures will be:

$$g_0(u) = f_0(u)[1 + \cos(2\pi b u)] \qquad (2)$$

where $b = B/D$ is the baseline normalised to the diameter D of a single circular aperture. The unit for the angular coordinate u is taken as λ/D where λ is the observed wavelength. The diffraction pattern of a single aperture is:

$$f_0(u) \propto \frac{2J_1^2(\pi u)}{(\pi u)^2}$$

baseline	1.0	1.5	2.5	3.5	4.5
$\sigma_{\rm p}$	4.7	3.3	2.0	1.4	1.1
$\sigma_{\rm h,opt}$	8.3	6.4	4.2	3.0	2.4

Table 1: Precision in the image position with optimum weighting function and photon noise, with $m_0 = 1000$. The first line is for an ideal detector, and the second for a finite pixel size and a sampling frequency twice the fringe frequency. The unit is nanorad for the nominal instrument (that is with $\lambda/D = 10^{-6}$).

In order to simplify the derived analytic expressions, $f_0(u)$ will be replaced with its gaussian approximation:

$$f_0(u) \propto \exp - \frac{u^2}{2\sigma_u^2} \tag{3}$$

with $\sigma_u = 0.432$

The precision in the position measurement of the interferogram with photon noise only is, with Eq. (1-3) :

$$\frac{1}{\sigma_p^2} = m_0 \left[\frac{1.0}{\sigma_u^2} + (2\pi b)^2 \right] \tag{4}$$

where m_0 is the number of photons collected with a single aperture of the interferometer. The coefficient 1.0 on the right hand side of the equation is in fact slightly dependent on the baseline, with departures from 1.00 smaller than a few 0.01.

The rms values σ_p of the position measurements are given in Table 1 for different baselines. With the nominal parameters for the instrument, $D = 0.55$ m and $\lambda = 0.55\mu$m, the unit for σ_p is a nanoradian.

For comparison, we evaluate the precision that should be reached with a full rectangular aperture containing the two circular apertures of the interferometer, that is with length $(b{+}1)D$ and width D. We obtain $\sigma_p = 1.7 \times 10^{-3}$ with $b = 2.5$ and the same photon flux. Compared to the value given for the interferometer, the improvement is less than expected from the increased number of photons due to the larger collecting area. We conclude that the Fizeau stellar interferometer is more efficient than a single aperture with respect to the precise location of the pattern in the image plane. This is due to the fine structures or fringes added in the diffraction pattern.

3. FILTERING AND SAMPLING WITH A REALISTIC DETECTOR

In the focal plane of the telescope, a CCD camera is supposed to operate in the time delayed integration mode. Its raws are aligned with the scanning direction, and we consider position measurements along this direction only. Let Δv the angular width of a pixel, and ω_0 the rotation speed of the spacecraft, the clock rate of the charge transfer, from one column to the next, will be

$$\Delta t = \Delta v / \omega_0$$

and the integration of photo-electrons is synchronized with the image displacement.

3.1. Filtering

The photon flux is summed both spatially and temporarily by the detector. In the $(v, \omega_0 t)$ plane, the cells have square shapes and signals from cells aligned with the first bisector are summed together. In the moving frame of the stellar image, the filtering function is the triangular function with base $2\Delta v$, the sampling frequency being $f_s = 1/\Delta v$. The filtering effect is easily evaluated in the Fourier space, the Fourier transform of the filter being the sinc function:

$$h(f) = \frac{\sin^2(\pi f/f_s)}{(\pi f/f_s)^2}$$

The interferometer fringes, at frequency $f_I = b$, have their amplitude multiplied by a coefficient $h_b = h(b)$. The observed interferogram will be approximated by:

$$g(u) \approx f_0(u)[1 + h_b \cos(2\pi b u)] \tag{5}$$

The precision in the position measurement of this smooth interferogram is estimated as in Section 2, that is with the method of the optimum weighting function and photon noise only. The precision is now given by:

$$\frac{1}{\sigma_h^2} = m_0 \left[\frac{1.0}{\sigma_u^2} + H(2\pi b)^2 \right] \tag{6}$$

with:

$$H = \frac{\int \mathrm{d}u A(u) f_0(u)}{\int \mathrm{d}u f_0(u)}$$

and:

$$A(u) = \frac{h_b + -\cos(2\pi b u)}{1 + h_b \cos(2\pi b u)}$$

The coefficient H is smaller than 1, except for $h_b = 1$. It is nearly vanishing for $h_b = 0$, that is for a sampling frequency equal to the fringe frequency.

3.2. Baseline Optimization

The sampling frequency is the ratio between the focal length of the telescope and the pixel width. With the nominal value $\lambda/D = 10^{-6}$ taken as unit, the sampling frequency is the ratio between the focal length, expressed in meters, and the pixel width in microns. Its value should be kept as small as possible, and certainly smaller than a few 10.

The sampling frequency being a strong constraint in direct fringe detection, we try to investigate what would be the optimum baseline for a given sampling frequency. Indeed, the coefficient H is a decreasing function of the baseline, and the precision in the position measurement given by Eq. (6) reaches a maximum for some intermediate baseline value. Numerical simulations with different sampling frequency ranging from 2 to 20 show that the optimum baseline is:

$$b_{\rm opt} \approx f_s/2 \tag{7}$$

For the optimum baseline value, the filter coefficient is $h_b = 0.405$, that is to say the fringes are appreciably smoothed out. The rms values $\sigma_{h,\rm opt}$ are also shown in Table 1 ; they have to be compared with the values σ_p for an ideal detector. For a given baseline, the ratio between these two values is about two. It increases slightly with the baseline. We conclude that the finite pixel width of a realistic detector brings quite a significant reduction in the precision of the position measurements.

4. CORRECTION FOR THE UNDERSAMPLING

Given the sampling frequency, the fringe frequency is half its value for the optimum baseline. The Nyquist rule for sampling is not satisfied, which means that the interferogram cannot be recovered exactly from its samples and the observed interferogram is dependent on the relative phase between the fringes and the samples. In the frequency (or Fourier) space, we find a case of spectrum folding quite similar to the problem of base band conversion nicely solved in radio astronomy (see Thompson et al. 1986). By analogy, we shall use two detectors (cf. two mixers) with a clock offset (cf. a phase shift of the local oscillator) and then two linear combinations of the outputs to separate positive and negative frequency bands of the sampled signal (cf. upper and lower sidebands).

Let $G(f)$ the Fourier transform of the interferogram $g(u)$. The sampling frequency f_s being approximately twice the fringe frequency, we can admit that

$$G(f) = 0 \text{ for } |f| \geq f_s$$

With $\nu = f/f_s$, the Fourier transform $\Gamma(\nu)$ of the undersampled interferogram is a periodic function with period equal 1. In the range $0 \leq \nu \leq 1$, the relation between Γ and G reduces to:

$$\Gamma(\nu) = G(\nu) + G(\nu - 1) \tag{8}$$

We suppose that the interferogram $g(u)$ is obtained with half a detector cell. Another half cell with a clock offset τ (expressed as a fraction of the clock rate Δt) will see a shifted interferogram, that is a linear phase shift $\phi = 2\pi\nu\tau$ in the Fourier space. With two half cells 1 and 2 and a clock offset $\tau = 1/2$ in between, what will be measured in the Fourier space is:

$$\Gamma_1(\nu) = G_1(\nu) + G_1(\nu - 1)$$

$$\Gamma_2(\nu) = G_2(\nu) + G_2(\nu - 1)$$

with:

$$G_2(\nu) = G_1(\nu)\mathrm{e}^{-i\pi\nu}$$

$$G_2(\nu - 1) = -G_1(\nu - 1)\mathrm{e}^{-i\pi\nu}$$

The two Fourier transforms of the undersampled interferograms are linearly combined to give:

$$\Gamma_+(\nu) = \Gamma_1(\nu) + \Gamma_2(\nu)\mathrm{e}^{i\pi\nu}$$

$$\Gamma_-(\nu) = \Gamma_1(\nu) - \Gamma_2(\nu)\mathrm{e}^{i\pi\nu}$$

and we verify:

$$\Gamma_+(\nu) = 2G_1(\nu)$$

$$\Gamma_-(\nu) = 2G_1(\nu - 1)$$

As $\nu \in [0, 1]$, the $(+)$ combination gives the unbiased Fourier transform of the interferogram for positive frequencies, and the $(-)$ combination gives the same for negative frequencies. The exact Fourier transform being recovered in the range $-1 \leq \nu \leq 1$, the result is the same as if the interferogram were sampled with the frequency $2 \times f_s$. The phase of the fringe train, and hence its location, is unaltered in this double detection process.

What about the signal-to-noise ratio? The pixel number is the same for each half cell, so that the signal-to-noise ratio (SNR) is *on average* the same for each detection, or

$$\overline{SNR_1} = \overline{SNR_2}$$

In the $(+)$ and $(-)$ combinations of two independent measurements, two Fourier components add in phase, and two other subtract, so that the signal-to-noise ratio for these two combinations is the same as the signal-to-noise ratio for each detection:

$$\overline{SNR_{+,-}} = \overline{SNR_{1,2}}$$

The noise fluctuations of the two combinations are uncorrelated *on average*, so that the resulting signal-to-noise ratio is $\sqrt{2}$ times the signal-to-noise ratio of a half cell measurement, that is the same as if a whole cell were used instead of the two half cells:

$$\overline{SNR_R} = \sqrt{2} \times \overline{SNR_{1,2}} \tag{9}$$

'On average' means for arbitrary phase between the fringes and the samples, and it may happen that the resulting signal-to-noise ratio is slightly increased or decreased relative to the *average result* owing to the phase of the individual measurements.

5. CONCLUSION

The precision of the image location has been estimated in the case of photon noise only. The effect of pixel width has been evaluated, and an optimum baseline found: the fringe frequency should be approximately half the sampling frequency. A method to correct for the effect of undersampling has been investigated. The method, with two adjacent half cells for the detector, requires that the diffraction pattern is unchanged in its shift from one to the other part of the detector. An alternative solution would be to have two photosensitive layers, and their associated CCDs at the same location in the focal plane, each layer detecting half of the incoming light, for example one and the other polarization.

A main step is now expected from the study of dedicated detector cells for direct fringe detection in the focal plane of the telescope.

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