THE TYCHO CATALOGUE: ASTROMETRY AND PHOTOMETRY

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ABSTRACT

The Tycho Catalogue of a million stars is described. The median value of the internal standard errors is $25~\mathrm{mas}$ for the position components and $0.10~\mathrm{mag}$ for the $B_T - V_T$ colour index. For stars brighter than $V_T = 9$ mag the median precision is 7 mas and 0.019 mag, respectively. The external astrometric standard errors have been determined by comparison with the Hipparcos Catalogue and are respectively 1.1 and 1.5 times larger for $V_T < 10$ mag and at $V_T = 10.5$ mag. The photon counts from the Hipparcos star mapper were the basic data used to derive the Tycho Catalogue. This large data set of about 100 Gigabytes could not be fully exploited in the first reduction. A second data reduction has therefore been started, based on the Tycho Catalogue and the experience gained, and utilizing more powerful computing facilities. This is expected to result in a second Tycho Catalogue of about 3 million stars.

Key words: space astrometry; Hipparcos; Tycho.

1. INTRODUCTION

The observational capabilities and operational principles of the Hipparcos satellite have been presented in the literature on previous occasions, with the most complete and definitive treatment contained within the published Hipparcos and Tycho Catalogues (ESA 1997). Short descriptions are given in this volume, and in Astronomy & Astrophysics by Perryman et al. (1997), Høg et al. (1997) and van Leeuwen et al. (1997).

Complete details of the data, the reductions, and the properties of the final Tycho Catalogue, are published by ESA (1997), especially in Volumes 1 and 4 to which we refer in the following. The contents of the catalogue is described in Section 4, and in more detail by Egret & Fabricius (1997).

The Tycho Catalogue of a million stars was obtained from photon counts obtained by scanning with the Hipparcos star mapper, carried out simultaneously with the Hipparcos observations in the adjacent main field of view of the telescope.

The key features of the Tycho observations (see also

Table 1) were very similar to those of the Hipparcos main field observations and may be summarised as follows:

(a) the continuous ecliptic-based scanning by the satellite, resulted in an optimum use of the available observing time, with a resulting catalogue of reasonably homogeneous limiting magnitude and astrometric accuracy;

(b) the various geometrical scan configurations for each star, at multiple epochs throughout the 3-year observation programme and simultaneously with the Hipparcos observations, resulted in a close connection to the Hipparcos astrometric reference frame represented by about 100 000 stars. The barycentric coordinate direction (α , δ), the parallax (π), and the object's proper motion ($\mu_{\alpha} \cos \delta$, μ_{δ}) could be solved for in least-squares reductions of the observations. These five 'astrometric' parameters as well as their standard errors and correlation coefficients were derived in the process;

(c) the large number of observations per object, of order 130, provided accurate and homogeneous photometric information for each star, from which mean magnitudes in two passbands, B_T and V_T were derived. Studies of variability and duplicity were undertaken.

The Tycho experiment also performed astrometric and photometric observations of solar system objects. These are described in the published Hipparcos and Tycho Catalogues (ESA 1997).

2. OVERVIEW

2.1. Astrometry

The five astrometric parameters, (α, δ) , the annual parallax, π , and the proper motion components $\mu_{\alpha*} = \mu_{\alpha} \cos \delta$ and μ_{δ} , in angular measure per unit time, are given for almost all stars in the catalogue. The catalogue includes astrometric quality flags, and indicators of variability and/or duplicity—nearly 900 000 of the catalogue entries are classified as 'recommended' reference stars, having good Tycho astrometric quality, and not recognised as double.

The median value of the formal (internal) astrometric standard errors given for each star (in position,

Measurement period	1989.85 - 1993.21
Catalogue epoch	J1991.25
Reference system	ICRS
coincidence with ICRS ¹	± 0.6 mas
deviation from inertial ¹	$\pm 0.25 \text{ mas/vr}$
Number of entries	1058332
based on Tycho data	1052031
with only Hipparcos data	6301
Median astrometric standard errors ²	
$V_T < 9 \text{ mag}$	7 mas
all stars	25 mas
Median photometric std. errors on V_T	
$V_T < 9$ mag	0.019 mag
all stars	0.06 mag
Sky density ³	$\sim 25 \text{ deg}^{-2}$
Limiting magnitude	$V \sim 11.5$
Completeness to ~ 90 per cent	$V\sim 10.5$
Completeness to ~ 99.9 per cent	$V\sim 10.0$
Total number of observations	$\sim 130 imes 10^6$

Table 1. Principal observational characteristics of the Tycho Catalogue. ICRS is the International Celestial Reference System.

¹ about all 3 axes;

² ratio of astrometric external to internal errors at $V_T = 10.5$ is ~ 1.5;

³ depending on galactic latitude.

parallax, and annual proper motion) are typically around 7 mas for stars brighter than $V_T \sim 9$ mag, and approximately 25 mas for $V_T \sim 10.5$ mag, The external standard errors (i.e. accuracies) are only slightly larger by a factor of 1.1 for bright stars of $V_T < 10$ mag, increasing to a factor of 1.5 larger at the median magnitude $V_T = 10.5$ of the Tycho Catalogue stars. This has been found from a comparison with the much more accurate Hipparcos values for stars common to the two catalogues.

Since for many Tycho Catalogue entries the proper motion has a standard error larger than the expected proper motion itself ($\mu \simeq 20 \text{ mas}/\text{yr}$ for a star of spectral type F5 and V = 11 mag), the accuracy of the Tycho proper motions is generally too low to calculate positions at other epochs with sufficient accuracy for reference purposes. More precise proper motions may be taken from the PPM Catalogue (Röser & Bastian 1991, Bastian et al. 1993, Röser et al. 1994) for approximately one half of the Tycho stars, although zonal errors present in the PPM should be considered in such applications—the Tycho Catalogue includes the PPM identifier where available. A plan for deriving proper motions for all Tycho stars by means of positions from the Astrographic Catalogue has been described by Röser & $H\phi g$ (1993) and Kuzmin et al. (1997). The accuracy of Tycho parallaxes is such that their astrophysical interest is restricted principally to the brighter stars (see Makarov 1997.)

Double stars were subject to dedicated data reductions, resulting in the resolution of pairs with separation down to about 1.5 arcsec. Detection of the effects of duplicity was effective down to separations of about 0.5 arcsec. The Tycho Catalogue contains various flags indicating such entries.

2.2. Photometry

The Tycho Catalogue includes accurate and homogeneous photometric information for each star: twocolour $(B_T \text{ and } V_T)$ magnitudes derived from the Tycho (star mapper) observations; Johnson V magnitude and B - V colour index, derived from the observed $B_T - V_T$ by simplified transformations reflecting the fact that the spectral type, luminosity class, reddening, etc., are unknown for the majority of Tycho Catalogue objects; and various flags resulting from (preliminary) studies of variability based on an average of 130 transits per star during the 3 year observing period. The details of the Tycho B_T and V_T photometric system, and details of the corresponding transformations to V and B-V, are given in the published catalogue, and also described in the context of the Hipparcos main mission photometry by van Leeuwen et al. (1997).

Published mean photometric values were based on all transits for each star, including those transits where the star was not detected because the signal was too faint. These 'censored' observations were taken into account in a dedicated 'de-censoring' processing. In the basic Tycho data reductions only the detections with a signal-to-noise ratio larger than 1.5 in the combined photon counts from the B_T and V_T detectors provided astrometric and photometric estimates. The photometric results for individual transits, including the censored observations, are provided separately in a Tycho Epoch Photometry Annex for specific subsets of stars. These individual magnitude determinations, at the specified measurements epochs, will allow detection of variable stars and further studies of the variability of known variables.

The principal photometric characteristics are as follows. Standard errors of V_T mean magnitudes are for $V_T < 9$ and ~ 10.5 respectively about 0.012 and 0.06 mag. Errors on B_T are typically 10 per cent larger for the same values of B_T . The standard errors of individual transits in the Tycho Epoch Photometry Annex are typically 10 times larger than for the mean value of a given star.

The distribution of all stars with at least one fully calibrated Tycho mean magnitude is shown in Figure 1 for B_T and V_T , i.e. for about 1 050 000 stars with median or de-censored magnitudes.

The catalogue is generally complete to 99.9 and 90 per cent at respectively V = 10.0 and 10.5 mag. But the degree of completeness at the faint magnitudes depends also on the colour index and on the position in the sky, due to the scanning law (see ESA 1997, Volume 4, Figures 16.4 and 16.5). The degree of completeness is approximately constant for all stars with a given value of $V_T = V_{T0} - 0.50(B_T - V_T)$ because the Tycho detection was carried out in the added photometric counts from the two channels B_T and V_T . This equation is almost equivalent to $V = V_0 - 0.70(B - V)$ where $V_0 = V_{T0}$, according to the transformation by Equations 1.3.20 in ESA (1997, Volume 1).

The distribution of the colour index B - V (Johnson system) is shown in Figure 2a, for all TYC stars with a value given for the colour index. Further diagrams show the distributions for four ranges

in galactic latitude b, but only for the stars with V < 11.0 - 0.70(B - V) mag. These stars have accurate values of B - V, i.e. σ_{B-V} less than about 0.10 mag. Some features related to the HR diagram of field stars in the solar neighbourhood are noted: the paucity of blue stars away from the galactic plane; a steep increase at B - V = 0.46 mag due to the main sequence turn-off; the peak around B - V = 1.0 due to the clump giants.



Figure 1. Distribution of the observed B_T and V_T for about 1 050 000 stars with fully calibrated magnitudes.



Figure 2. Distribution of the colour index B - V in the Johnson system, derived from the observed B_T and V_T . (See explanation in Section 2.2)

3. CATALOGUE PRODUCTS

The designation of objects within the Tycho catalogue uses the Guide Star Catalog (GSC) numbering system (a region number, designated TYC1; and a number within the region, designated TYC2), followed by a Tycho Catalogue specific component number (TYC3), the latter introduced to permit the parallel classification of resolved systems. As well as giving a cross-identification to the GSC, this designation system has the advantage of giving a rough indication of position on the sky. For objects contained in the GSC, TYC1 and TYC2 are identical to the identifiers defined by the GSC numbering system. Cross-references are given to the Hipparcos Catalogue identifiers for 123 431 entries, with cross-identifications also given to the CCDM, PPM, HD and DM (BD/CoD/CPD) catalogues, and a flag pointing to the variable star catalogues GCVS and NSV, based on the CDS SIMBAD facility.

The Tycho Catalogue is available as part of the 17volume publication covering the Hipparcos and Tycho Catalogues (ESA 1997), Volume 1 includes descriptions of the various data sets of the Tycho (and Hipparcos) Catalogue, and provides statistics and sky distributions of the major catalogue parameters. Volume 4 describes the construction of the Tycho Catalogue. In Volume 17, CD-ROM 1 contains the main Tycho Catalogue, comprising 57 fields for each star record covering the astrometric results, mean photometry, flags, and cross-identifications. CD-ROM 4 contains the Tycho Epoch Photometry Annex A with transit data for 34 446 objects. Epoch photometry for a significantly larger fraction of the Tycho Catalogue objects (481 553, including those in Annex A) is made available as the Tycho Epoch Pho-tometry Annex B through the CDS, Strasbourg. The complete epoch photometry data for all Tycho Catalogue objects are kept in an archive of the Tycho Consortium.

4. CONTENTS OF THE TYCHO CATALOGUE

Unlike the main Hipparcos Catalogue, the Tycho Catalogue and the Tycho Epoch Photometry Annex are not made available in printed form, but only as a machine-readable version.

The Tycho Catalogue format is largely identical to that of the Hipparcos Catalogue up to and including Field H39/T39. This approach is intended to facilitate use of both catalogues. Thereafter, additional Tycho Catalogue fields are included with astrometric and photometric information and crossidentifications to other catalogues. Flags indicate variable stars, detected duplicity or multiplicity, or notes.

Entries of the Tycho Catalogue are given in the sequence of the hierarchical TYC1-3 identifier (Field T1), i.e. corresponding to organisation according to the GSC region number (and not in sequence of right ascension).

The same conventions and units are adopted for the Tycho Catalogue as for the main Hipparcos Catalogue. Thus, definitive values of right ascension and declination are expressed in degrees and decimals of a degree, while all other astrometric parameters, and errors, are expressed in milliarcsec, even though some digits may not be significant. The catalogue epoch is J1991.25, and positions and errors are given for that epoch. Astrometric correlations are also provided. All parameters are expressed within the reference system ICRS.

Double and multiple star detection differs significantly between the Hipparcos main mission and Tycho, and fields dedicated to summary double and multiple star data in the main Hipparcos Catalogue are accordingly absent in the Tycho Catalogue.

The following brief description of the fields summarizes, but can not replace, the detailed description in Volume 1, Section 2.2 of ESA (1997). Erroneous or misleading conclusions might be avoided by proper use of the flags, especially those emphasized by italics in the following listing: T10, T36, T40, T42, T57. It is noted that due care must be taken in ensuring that blank fields are not interpreted as numerically zero.

4.1. Fields T1–39, Similar to Fields in HIP

Field T1. TYC1-3, the TYC identifier.

Field T2. A 'proximity flag' derived from nearby HIP and TYC entries; the flag indicates that caution is needed in using the star as an astrometric reference, and this is taken into account in the definition of the flag T10.

Fields T3–4. Sexagesimal identifier: provided 'for information' and derived from the definitive position. The epoch is J1991.25.

Field T5. Johnson V magnitude.

Field T6. This field is blank for Tycho.

Field T7. Source of magnitude given in Field T5.

Field T8–9. The Tycho position, at epoch J1991.25.

Field T10. Reference flag for the astrometric parameters. A 'blank' indicates that the astrometric parameters refer to one of the 886 621 'recommended' astrometric reference stars, having good Tycho astrometric quality, and not recognised as double.

Fields T11–13. Trigonometric parallax (mas) and two proper motion components (mas/yr).

Fields T14–18. Standard errors of the five astrometric parameters.

Fields T19-28. Correlation coefficients of the astrometric parameters.

Field T29. The number of detected transits, N_{astrom} , retained in the astrometric adjustment.

Field T30. Goodness-of-fit parameter; should be used with caution.

Field T31. The Hipparcos Catalogue number for the star, if contained in HIP.

Fields T32–35. The B_T mean magnitude and standard error σ_{B_T} . The V_T mean magnitude and standard error σ_{V_T} .

Field T36. Source of photometric data. Over one million stars obtained de-censored mean magnitudes, about 30 000 bright stars obtained median magnitudes. For about 4600 stars only approximate magnitudes could be derived.

Field T37–38. The B-V colour index, and standard error. The value of B-V gives an indication of the Johnson B-V colour index. It is derived strictly from the B_T and V_T in the preceding fields by an approximate transformation.

Field T39. Blank for Tycho.

4.2. Fields T40-57, Tycho-Specific Fields

Field T40. Astrometric quality flag, Q, on a scale from 1–9 where 1 is the best quality (see Section 5).

Field T41. Signal-to-noise ratio of the star image, F_s . The value, always > 3.0, was used in constructing the Q in the preceding field.

Field T42. Source of astrometric and other data. The field is non-blank for about 23 000 stars for which the standard Tycho processing did not produce adequate results, so that data were taken from other sources as indicated by the value of the flag.

Field T43. The number of transits, $N_{\rm photom}$, used in Tycho mean photometry.

Field T44. The V_T scatter, derived from the distribution function of the individual observations of V_T .

Fields T45–46. The V_T magnitudes at respectively maximum and minimum luminosity.

Field T47. Flag indicating previously known or suspected variable.

Field T48. Flag indicating variability of the Tycho photometric measurements.

Field T49. Flag indicating unresolved duplicity status from the Tycho data analysis.

Field T50. Flag indicating availability of epoch photometry for this object.

Field T51. The CCDM component identifier, required for the unique identification of a double or multiple star component in Part C of the Hipparcos Double and Multiple Systems Annex.

Field T52. The PPM identifier. The PPM catalogue is the largest source to date for high-precision proper motions, containing a very significant fraction (about 42 per cent) of the Tycho Catalogue stars.

Field T53–56. HD/HDE/HDEC identifier. DM identifier (BD). DM identifier (CoD). DM identifier (CPD).

Field T57. Notes on dubious entries. The flag is non-blank for 36 717 entries, indicating some kind of

Table 2. The astrometric quality Q and associated quantities. N is the number of stars in the Tycho Catalogue of each quality class. σ_{med} gives the median standard error of a position coordinate at the catalogue epoch J1991.25 (the errors at the mean epoch of observation of any given star are typically 5 per cent smaller).

Q	$\sigma_{\rm max}$ (mas)	F_s	$\sigma_{\rm obsf}$ (mas)	Ν	$\sigma_{\rm med}$ (mas)	Astrometric
	(mab)		(mas)		(mas)	quanty
1	< 5	> 5	< 300	23147	2.6	very high
2	5 - 10	> 5	< 300	70945	5.5	very high
3	10 - 25	> 5	< 300	259695	13	high
4	25 - 50	> 5	< 300	430182	26	high
5	50 - 150	> 5	< 300	146520	39	medium
6	< 150	> 5	≥ 300	41695	44	perhaps non-single
7	< 150	3 - 5	< 300	37821	45	low
8	< 150	3 - 5	≥ 300	28949	54	perhaps non-stellar
9	$\equiv 200$	-	-	13077	-	low, 'R' in Field T42
Ц	-	-	-	6301	-	unassigned, 'H' in Field T42
any	_	=	_	1058332	25	all entries

discrepancy of astrometric or photometric data, either internally for Tycho data, or by comparison of Tycho values with other sources of data, namely the Guide Star Catalog or PPM.

5. ASTROMETRIC QUALITY

The standard errors of the five astrometric parameters are good quality indicators for the solution for bright stars. But they were not sufficient for the present purpose, a reliable distinction of good solutions from bad ones, especially for the faint stars where the single transit could have a signal-to-noise ratio of only 1.5. Special indicators were required to measure the total signal-to-noise ratio of the combined signals from the whole mission in order to distinguish between a real star on the sky and a 'false' star, generated by a random accumulation of noise.

The final astrometric quality flag Q is mainly based on the three quality indicators σ_{max} , F_s and σ_{obsf} (columns 2–4 in Table 2).

- The Indicator σ_{max} : This quantity is the largest of any of the five astrometric standard errors for a given star.
- Signal-to-Noise Ratio, F_s : This quantity was based on an examination of the distribution of astrometric residuals (see ESA (1997), Volume 4, Chapter 7) and was computed in a separate processing of all observations after the final astrometric parameters had already been computed. The quantity is defined as:

$$F_s = (n_1 - n_2) / \sqrt{n_1 + n_2} \tag{1}$$

where n_1 is the number of detections within 0.7 arcsec along scan from the mean position, and n_2 is the number of detections between 0.7 and 1.4 arcsec (which provides a measure of the rate of background detections). In the special but rather common case of a sharp image on a negligible background, $n_1 \sim N_{\rm astrom}$, $n_2 \sim 0$ and

 $F_s \sim \sqrt{N_{\rm astrom}}$ where $N_{\rm astrom}$ is the number of accepted astrometric detections.

• Half-Width of the Image: σ_{obsf} is the formal standard error of the single observation: $\sigma_{obsf}^2 = 0.25 (\sigma_x^2 + \sigma_y^2) (N_{astrom} - 5)$, where σ_x and σ_y are the standard errors of the two position components. It is a measure of the half-width of the observed star image. A half-width larger than 300 mas was found to be correlated with duplicity or a non-stellar image.

5.1. Astrometric Quality Flag

The astrometric quality, Q, is defined for the Tycho data according to Table 2, where N gives the number of stars in the Tycho Catalogue of each quality class. Objects with $Q \leq 8$ in the Tycho Catalogue all have $F_s > 3$, $\sigma_{\max} < 150 \text{ mas}$, $\sigma_{obsf} < 450 \text{ mas}$, and $N_{astrom} > 30$. The last criterion, $N_{astrom} \leq 30$, had only the effect of excluding less than 100 stars of low quality, due to the F_s limit being the strongest criterion.

Objects with Q = 9 have lower astrometric quality, and are included for the sake of their photometric data—these objects are flagged by '*R*' in Field T42 (ESA (1997), Section 2.2, Volume 1). The last line in the table shows that 6301 entries have '*H*' in Field T42, showing that they are contained in the Hipparcos Catalogue, but were not observed by Tycho.

A flag is provided (in Field T10) showing if a star is a dubious astrometric reference star, e.g. due to suspected duplicity or dubious astrometry. The remaining approximately 900 000 stars are called 'recommended astrometric reference stars'.

5.2. Verification of Quality

Development of the quality criteria and their proper combination into astrometric quality classes was a long process where various tools were used. Local sky maps of the actual Tycho catalogue were compared with direct photographs or with the Digitized Sky Survey (DSS). The DSS happened to become available just as needed and it was very extensively used to distinguish stars and nebulae on the sky and compare with entries of various quality classes and false stars. Comparison with the GSC 1.1 was also much used. Pair statistics were used, similar to Figure 3. Distribution functions of various quantities were plotted for selected subsets of data as shown by the example in ESA (1997), Figure 7.6 of Volume 4.



Figure 3. Distribution of distances between all pairs of Tycho Catalogue stars. The dotted line corresponds to a uniform (random) distribution of the same number of stars on the sky. The peaks at 5.6, 11, 17, 23, 29 and 34 arcsec are due to side lobes of the response function of the Tycho star mapper slits. A star may give rise to false detections at these distances or make true detections brighter when another star is crossing the slit system at the same time.

6. THE SECOND TYCHO PROCESSING

All processing of Tycho observations used for the Tycho Catalogue described above was based on detection of the stars at each crossing of the star mapper above a certain signal-to-noise ratio (1.5 or 1.8).

Further improvement of limiting magnitude by about 0.4 mag could be obtained by superposition of the photon counts for each star from two consecutive observations in the preceding and following fields of view of the Hipparcos telescope. This was beyond capabilities with respect to software development and computing facilities when discussed earlier (in 1991).

A photon superposition for the whole mission would give a still higher gain, and is now being carried out at Copenhagen, in a collaboration with scientists at Heidelberg. The completion is expected in 1999. This idea was proposed in principle by Høg et al. (1982), but had to be abandoned in favour of the more modest approach used for the present Tycho Catalogue. The second Tycho processing of all raw photon counts will be based on the available satellite attitude, the Tycho Catalogue of one million stars, and an input catalogue of about 6 million stars from a new reduction of GSC and the Astrographic Catalogue so that a position accuracy about 0.25 arcsec at the observation epoch is provided. This will greatly facilitate the processing which is expected to give better astrometric and photometric values for the fainter half of the present one million stars. The major reason is that an astrometric estimation based on very few photons does not achieve the Cramér-Rao limit, see Yoshizawa, Andreasen& Høg (1985). An estimation based on all photons from many transits will come much closer to that limit. Furthermore, many transits of the faint stars were below the limit of signalto-noise ratio for the detection and were thus completely lost. Good results are expected for altogether up to $3\,000\,000$ stars, brighter than about V = 12. Expected external standard errors at V = 11.0 and 12.0 mag are respectively $\sigma_{\alpha,\delta} \simeq 50$ and 100 mas and $\sigma_V \simeq 0.10$ and 0.20 mag.

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