STATISTICAL ASTROMETRY BASED ON A COMPARISON OF INDIVIDUAL PROPER MOTIONS AND POSITIONS OF STARS IN THE FK5 AND IN THE HIPPARCOS CATALOGUE

R. Wielen, H. Schwan, C. Dettbarn, H. Jahreiß, H. Lenhardt

Astronomisches Rechen-Institut, Mönchhofstraße 12-14, D-69120 Heidelberg, Germany

ABSTRACT

The wavy motions of the photocenters of undetected astrometric binaries in a sample of apparently single stars introduce 'cosmic errors' into high-precision astrometry, especially into 'instantaneously' measured proper motions and positions. For bright Hipparcos stars, we derive from a comparison of their proper motions given in the basic FK5 and in the Hipparcos Catalogue that the typical cosmic error in the Hipparcos proper motions of these stars is about 2 mas/year. Hence the cosmic errors are about three times larger than the measuring errors of the Hipparcos proper motions for bright stars (typically better than 0.7 mas/year), and they should therefore be taken into account in applications of the Hipparcos data. The fact that astrometric binaries are actually the most important cause of the cosmic errors are proved by the decrease of the cosmic errors with increasing distances of the stars and by the large cosmic errors in a sample of astrometric binaries which are identified as such by Hipparcos because of their non-linear motions (stars with significant g or $\dot{g}$ acceleration terms). For a consistent treatment of cosmic errors in high-precision astrometry, the coherent scheme of 'statistical astrometry' has been developed and is shortly summarized here. We discuss some implications of the presence of cosmic errors in the instantaneously measured Hipparcos data, and we outline the project of the FK6 where we combine the ground-based results of the basic FK5 with the Hipparcos data in order to obtain proper motions and positions of highest accuracy, taking the concepts of statistical astrometry into account.

Key words: statistical astrometry; cosmic errors; Hipparcos Catalogue; FK5 Catalogue; FK6 Catalogue.

1. INTRODUCTION

It is probably one of the most remarkable results of the Hipparcos mission that we are now, for the first time, able to obtain quantitative information about 'cosmic errors' in 'instantaneously' measured proper motions and positions of stars. The standard model, used in most astrometric data-reduction procedures, assumes a linear motion of a star in time and space, as it is very appropriate for single stars. This standard model is also adopted for most of the stars in the Hipparcos Catalogue (with the addition of the effect of the parallax).

From ground-based observations of stars, however, it is well-known that the vast majority of objects are in fact binaries (e.g. Duquennoy and Mayor 1991). Hipparcos has detected about 10 per cent of the observed objects directly as visual binaries. The remaining 90 per cent of the objects look like a single star for Hipparcos. Only for a small percentage (about 4 per cent) of these seemingly single stars, Hipparcos was able to detect the binary nature of the objects by measuring a non-linear motion of the photo-center, e.g. a Keplerian orbit (given as an 'orbital solution' in the Double and Multiple Systems Annex of the Hipparcos Catalogue), or a curved motion (acceleration solutions with so-called g or $\dot{g}$ terms), or by other indications for duplicity (stochastic solutions, VIM solutions). Hence a large fraction of the apparently single stars must be actually undetected and unresolved binaries.

Even if Hipparcos was unable to detect direct evidence for the duplicity of an object during its observational period of about three years, the Hipparcos proper motions and positions allow us to obtain indirect information on the binarity of objects by comparing these results with other data, e.g. with ground-based proper motions.

The astrometric-binary nature of many apparently single stars causes deviations from an assumed linear motion. In Figure 1, it is illustrated that an 'instantaneously' measured proper motion depends on time and deviates also in general from the proper motion of the center-of-mass, i.e. from the long-term average of the proper motion. We have proposed to call such deviations in proper motions and positions from the mean motions 'cosmic errors'. Clearly, the cosmic errors are actually a defect of our standard model. However, terming the deviations 'modelling errors' seems inappropriate to us, because such a term may suggest that we could actually do better, which is not true at present for most of these apparently single stars.

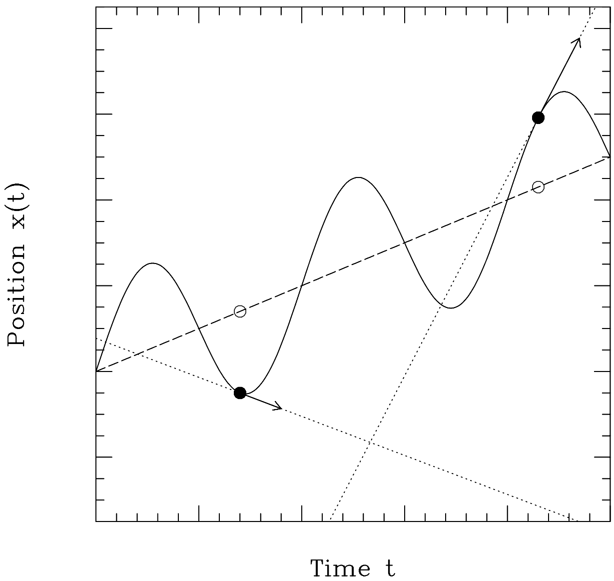

Figure 1. Wavy motion of the photo-center of an astrometric binary (solid curve) around the linear motion of its center-of-mass (dashed line). Two instantaneous positions and proper motions are indicated (filled dots; arrows). The linear predictions based on the instantaneous values are shown as dotted lines.

The effect of undetected astrometric binaries on high-precision astrometric measurements, such as obtained by Hipparcos, has been already discussed theoretically by Lindegren (1979), Soederhjelm (1985), Tokovinin (1993), Brosche et al. (1995), and Wielen (1995 a, b, 1996, 1997). Here we give now actual observational results on such 'cosmic errors' by comparing the Hipparcos measurements with the data given in the basic FK5.

2. COMPARISON OF THE HIPPARCOS CATALOGUE WITH THE BASIC FK5

In order to obtain reliable information on the cosmic errors in astrometric data, we should obviously use the most accurate data. This leads to a sample which consists of the basic FK5 stars. Among the ground-based results, the proper motions and positions of the basic FK5 stars are by far the most accurate ones. The basic FK5 stars belong to the brighter stars, with visual magnitudes V mostly brighter than 7.5 mag. These bright stars have also the smallest measuring errors in the Hipparcos Catalogue. From all the 1535 basic FK5 stars, we use only a subsample of 1202 apparently single stars. All these apparently single stars are carefully checked for not showing any indication of a disturbing duplicity from either ground-based observations or from Hipparcos data.

Before we compare the FK5 data (Fricke et al. 1988) with the Hipparcos results (ESA 1997), we first eliminate the systematic deviations in positions and proper motions between the FK5 and Hipparcos by reducing the FK5 to the Hipparcos system. This can be done with sufficient accuracy, as will be discussed in Section 2.2.

2.1. Global Comparison of Proper Motions

For the subsample of the 1202 apparently single stars, we compute the differences in their proper motions, given in the basic FK5 (μ_F) and in the Hipparcos Catalogue (μ_H), separately for $\mu_{\alpha\cos\delta}$ and μ_δ. For describing these differences statistically, we use the dispersion D_μ of the differences, i.e. their root-mean-squared (rms) value:

$$\begin{aligned} D_{\mu,\mathrm{obs}} &= \Big(\frac{1}{2}\,\big(< (\mu_{F,\alpha\cos\delta} - \mu_{H,\alpha\cos\delta})^2 > \\ &\qquad + < (\mu_{F,\delta} - \mu_{H,\delta})^2 > \big)\Big)^{1/2} \\ &= 2.38 \text{ mas/year} \end{aligned} \tag{1}$$

From the rms values of the individual measuring errors of the proper motions in the Hipparcos Catalogue (0.68 mas/year), in the FK5 (0.77 mas/year), and of the mean errors of the systematic corrections (0.28 mas/year), we should expect a value $D_{\mu,\mathrm{exp}}$, if there were no cosmic errors, i.e. for truly single stars:

$$D_{\mu,\mathrm{exp}} = 1.07 \text{ mas/year} \tag{2}$$

The quadratic difference between $D_{\mu,\mathrm{obs}}$ and $D_{\mu,\mathrm{exp}}$ is the contribution $D_{\mu,\mathrm{cosm}}$ of the cosmic errors in the proper motions to $D_{\mu,\mathrm{obs}}$:

$$\begin{aligned} D_{\mu,\mathrm{cosm}} &= (D^2_{\mu,\mathrm{obs}} - D^2_{\mu,\mathrm{exp}})^{1/2} \\ &= 2.13 \text{ mas/year} \\ &\approx 3 \cdot \text{measuring error of the} \\ &\qquad \text{Hipparcos proper motions} \end{aligned} \tag{3}$$

The rms value of the cosmic errors, which are expected to occur mainly in the instantaneously measured Hipparcos proper motions, is found to be about three times larger than the measuring error of the Hipparcos proper motions ! The cosmic errors in the Hipparcos proper motions are therefore quite significant for the basic FK5 stars and cannot be neglected in many applications.

For describing the cosmic errors statistically, we have used their dispersion D (see Equation 1). However, the distribution function of the cosmic errors is certainly far from being gaussian. It is strongly peaked around zero, with extended wings, and contains also truly single stars with no cosmic errors at all. This means that only a minority of apparently single stars are strongly affected by cosmic errors, while the majority is only weakly affected (or not at all). Nevertheless, the dispersion D of the cosmic errors is the most direct and simplest measure of the cosmic errors in general. Unfortunately, it is usually not possible to exclude a priori those apparently single stars from our astrometric work which have by chance large cosmic errors. They can usually be only identified a posteriori, if at all, e.g. as being 'outliers'. As discussed below, the best way to keep the cosmic errors small is to use stars with very large distances from the Sun.

2.2. Variation of the Cosmic Errors with Stellar Distances

A sceptic may fear that the cosmic errors are not real but only formally caused by having under-estimated the actual measuring errors of the FK5 or the Hipparcos Catalogue or both. This possibility is largely ruled out, however, by the dependence of the cosmic errors on the distance of the stars.

Table 1. Cosmic errors in proper motions as a function of the parallax.

	Parallax p [mas]				
	all	≥ 20	20–10	10–5	< 5
$D_{\mu,\mathrm{obs}}$ [mas/year]	2.38	2.94	2.50	2.05	1.58
$D_{\mu,\mathrm{exp}}$ [mas/year]	1.07	1.01	1.06	1.11	1.11
$D_{\mu,\mathrm{cosm}}$ [mas/year]	2.13	2.76	2.26	1.72	1.12
median of r [pc]	91	30	72	136	310
median of p [mas]	11	33	14	7	3
$D_{\mu,\mathrm{cosm}}$ [km/s]	0.92	0.40	0.78	1.10	1.64

In Table 1, we list the cosmic errors for four groups formed according to the distances r of the stars from the Sun. Most of the stellar distances are obtained from the trigonometric parallaxes given in the Hipparcos Catalogue. Table 1 shows that the cosmic error $D_{\mu,\mathrm{cosm}}$ decreases strongly with increasing distance r (i.e. with decreasing parallax p) of the stars. This is to be expected for the case that the cosmic errors are due to unresolved astrometric binaries, because of the shrinking of the binary effects in an angular measure with increasing distance of the object. The fact that the cosmic error measured in absolute units (e.g. in km/s) is increasing with distance r (instead of being constant) is also quite plausible, since the fraction of undetected binaries is increasing with increasing r, and because the absolute luminosities of the stars (and hence the stellar masses and the binary effects) are absolutely larger at larger distances for a magnitude-limited sample like the basic FK5.

The dependence of the cosmic error on the stellar distance r is also an indirect proof for the reliability of the systematic corrections applied (see Section 2), since large errors in these corrections, which are indepent of r, would largely erase the dependence of the cosmic errors on r.

2.3. Comparison of Positions

In Table 2, we compare the positions of stars, given by the FK5 and by the Hipparcos Catalogue, at two different epochs T, namely at the mean epoch of the FK5, $T_F \sim 1950$, and of Hipparcos, $T_H \sim 1991.25$. For obtaining the FK5 positions at T_H, we use the FK5 positions at their central epochs T_F and the FK5 proper motions for the epoch transformation. Similarly, the Hipparcos positions at T_F are based on the Hipparcos positions at T_H and the Hipparcos proper motions. For a statistical description of the differences of the positions, we use the dispersion of these differences, $D_{\mathrm{pos,obs}}(T)$, defined in analogy to Equation 1. The 'expected' value $D_{\mathrm{pos,exp}}$ is calculated from the measuring errors in the FK5 and Hipparcos positions at their central epochs, and from the measuring errors of the proper motions used, either μ_H or μ_F, and the individual epoch differences. The quadratic difference (see Equation 3) between $D_{\mathrm{pos,obs}}(T)$ and $D_{\mathrm{pos,exp}}(T)$, is called $D_{\mathrm{pos,cosm}}(T)$, and characterizes the effect of the cosmic errors on the positions.

Table 2. Comparison of positions in the FK5 and in the Hipparcos Catalogue.

at epoch:	$T_F \sim 1950$	$T_H \sim 1991.25$
using:	μ_H	μ_F
$D_{\mathrm{pos,obs}}$ [mas]	85.5	59.6
$D_{\mathrm{pos,exp}}$ [mas]	36.1	38.6
$D_{\mathrm{pos,cosm}}$ [mas]	77.5	45.4

From Table 2, we see that the cosmic error contribution on $D_{\mathrm{pos,cosm}}$ is about twice as large at T_F than at T_H. Since $D_{\mathrm{pos,cosm}}(T_F)$ is using μ_H, but not μ_F, and since measuring uncertainties in the two central positions (HIP and FK5) affect $D_{\mathrm{pos,cosm}}$ at T_H and T_F by the same amount, Table 2 shows that the cosmic error in the Hipparcos proper motions is significantly larger than in the FK5. This is to be expected a priori: The Hipparcos proper motions are 'instantaneously' measured and are therefore strongly affected by binary motions. In contrast, the FK5 proper motions are 'mean' values, based on positions obtained over more than two centuries. Hence the FK5 proper motions are much closer to the center-of-mass motions and hence to a 'long-term average'.

The quantities $D_{\mathrm{pos,cosm}}$ at T_F and T_H are governed by the cosmic errors in the proper motions. Hence it is difficult to obtain a reliable observational estimate of the cosmic errors in the instantaneously measured positions given in the Hipparcos Catalogue. From other, statitistical considerations we expect these cosmic errors to be of the order of 10 mas for the basic FK5 stars. This would mean that the cosmic errors in the Hipparcos positions of these stars are much larger than their measuring errors.

2.4. Comparison for Stars with Individually Significant Acceleration Terms

In order to confirm that the cosmic errors are actually due to astrometric-binary motions, we repeat the comparisons presented in the former sections for a group of basic FK5 stars which are already safely established as being astrometric binaries. These are 95 basic FK5 stars listed in the G part of the Double and Multiple Systems Annex of the Hipparcos Catalogue. Stars in this G Annex have individually significant deviations from the linear motion in form of acceleration terms (g or $\dot{g}$ terms). These 95 stars from the G Annex were, of course, not included into our subsample of 1202 apparently single stars. For these stars from the G Annex, we use the proper motions and positions given in the main part of the Hipparcos Catalogue. Hence these data are 'averaged' values over the observational period of Hipparcos.

Table 3. Cosmic errors in the proper motions and positions of stars with acceleration solutions (G Annex; $g + \dot{g}$ stars).

Sample:	$g + \dot{g}$ stars	'single' stars
number (% of basic FK5)	95 (6%)	1202 (78%)
$D_{\mu,\mathrm{obs}}$ [mas/year]	9.33	2.38
$D_{\mu,\mathrm{exp}}$ [mas/year]	1.31	1.07
$D_{\mu,\mathrm{cosm}}$ [mas/year]	9.24	2.13
$D_{\mathrm{pos,obs}}(T_F)$ [mas]	391.3	85.5
$D_{\mathrm{pos,exp}}(T_F)$ [mas]	48.9	36.1
$D_{\mathrm{pos,cosm}}(T_F)$ [mas]	388.2	77.5
$D_{\mathrm{pos,obs}}(T_H)$ [mas]	71.5	59.6
$D_{\mathrm{pos,exp}}(T_H)$ [mas]	39.2	38.6
$D_{\mathrm{pos,cosm}}(T_H)$ [mas]	59.8	45.4

Table 3 shows that the cosmic errors for the stars with G solutions are extremely well established. This confirms that the cosmic errors are actually due to astrometric binaries. The fact that the cosmic errors are much larger for the stars with G solutions than for apparently single stars is to be expected, since the latter sample contains certainly actually single stars and since the binaries in this sample are less 'active', because Hipparcos was not able to identify them as astrometric binaries individually. The fact that $D_{\mathrm{pos,cosm}}$ is much larger at T_F than at T_H, confirms our statement in Section 2.3 that the cosmic errors mainly affect the 'instantaneous' Hipparcos proper motions and not so much the 'mean' FK5 proper motions.

3. STATISTICAL ASTROMETRY

While our comparisons of the proper motions and positions of stars given in the basic FK5 and in the Hipparcos Catalogue, presented in the former sections, show clearly the existence of cosmic errors in high-precision astrometric data and give first quantitative results on the amount of such cosmic errors, the correct treatment of the effects of cosmic errors on astrometric results requires a new framework: 'statistical astrometry'. It is the aim of statistical astrometry to provide a coherent scheme, and methods and tools, for high-precision astrometric measurements which are affected by cosmic errors. The principles of statistical astrometry are presented by Wielen (1997). Here, we can only provide an extremely short summary of statistical astrometry.

The basic tools of statistical astrometry are correlation functions which describe the statistical correlations between orbital displacements in position, Δx, and in velocity, Δv, at different epochs of time t. The orbital displacements are measured relative to the mean motion of the object. The most important correlation functions are:

$$\xi(\Delta t) \quad = \quad < \Delta x(t + \Delta t)\, \Delta x(t) > \tag{4}$$

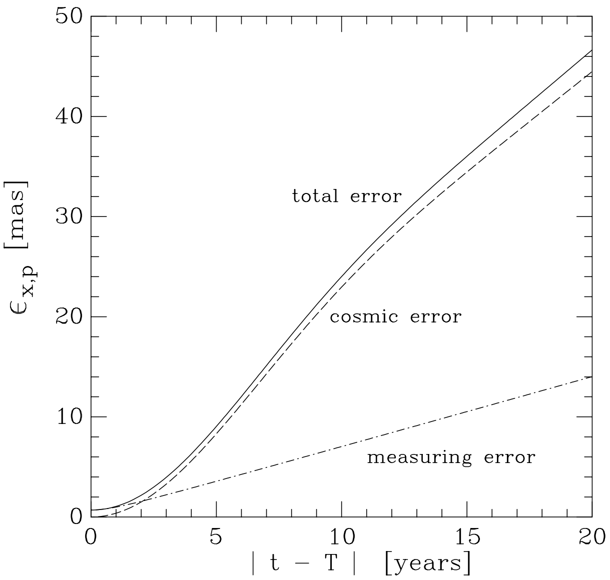

Figure 2. Mean error $\varepsilon_{x,p}(t)$ of a predicted instantaneous position $x_p(t)$ at an epoch t, based on a linear extrapolation using an instantaneous position and an instantaneous proper motion observed at epoch T. The total mean error (solid curve) is built up by the cosmic error (dashed curve) and the measuring error (dash-dotted curve).

$$\eta(\Delta t) \quad = \quad < \Delta v(t + \Delta t)\, \Delta v(t) > \tag{5}$$

$$\zeta(\Delta t) \quad = \quad < \Delta x(t + \Delta t)\, \Delta v(t) > \tag{6}$$

The average is taken over the sample of stars under consideration and over the time t. Hence the correlation functions depend on the epoch difference Δt only.

A typical problem of statistical astrometry is the following: Let us assume that we have measured at an epoch T the instantaneous position $x(T)$ and the instantaneous velocity (or proper motion) $v(T)$ of a star. The measuring errors of $x(T)$ and $v(T)$ are $\varepsilon_{x,\mathrm{meas}}(T)$ and $\varepsilon_{v,\mathrm{meas}}(T)$. We are now interested in predicting an instantaneous position $x(T)$ at another epoch t. If we have no other information beside $x(T)$ and $v(T)$, we can only use a linear extrapolation $x_p(t)$ as the best estimator for $x(t)$:

$$x_p(t) \quad = \quad x(T) + v(T) \cdot (t - T) \tag{7}$$

How large is the expected uncertainty $\varepsilon_{x,p}(t)$ in $x_p(t)$? In the classical case of a single star, we would have to consider the uncertainty $\varepsilon_{x,p,\mathrm{meas}}(t)$ due to the measuring errors only:

$$\varepsilon^2_{x,p,\mathrm{meas}}(t) = \varepsilon^2_{x,\mathrm{meas}}(T) + \varepsilon^2_{v,\mathrm{meas}}(T) \cdot (t - T)^2 \tag{8}$$

In the case of the presence of cosmic errors in our sample, which then contains undetected astrometric binaries, we have to consider the cosmic-error contribution $\varepsilon_{x,p,\mathrm{cosm}}(t)$ also. In statistical astrometry it is shown (Wielen 1997) that the cosmic error $\varepsilon_{x,p,\mathrm{cosm}}(t)$ can be derived from the correlation functions by:

$$\varepsilon^2_{x,p,\mathrm{cosm}}(t) = 2\,(\xi(0) - \xi(t - T))$$
$$-2\,\zeta(t - T) \cdot (t - T) + \eta(0) \cdot (t - T)^2 \tag{9}$$

The total expected uncertainty $\varepsilon_{x,p,\mathrm{tot}}(t)$ in $x_p(t)$ is then given by

$$\varepsilon^2_{x,p,\mathrm{tot}}(t) = \varepsilon^2_{x,p,\mathrm{meas}}(t) + \varepsilon^2_{x,p,\mathrm{cosm}}(t) \qquad (10)$$

since the measuring errors are usually not correlated with the cosmic errors.

In Figure 2, we apply the results presented above to a case which is probably rather representative for bright Hipparcos stars. The measuring errors are assumed to be $\varepsilon_{x,\mathrm{meas}}(T) = 0.7$ mas and $\varepsilon_{v,\mathrm{meas}}(T) = 0.7$ mas/year. The correlation functions used are described by Wielen (1997). The most important quantity for the cosmic errors is $(\eta(0))^{1/2}$, for which the value 2.11 mas/year is used in our example. This is quite close to our results for cosmic errors in Section 2 (see Equation 3, for example). Figure 2 shows clearly that the cosmic errors are more important for the uncertainty of our predicted position $x_p(t)$ than the measuring error, if the epoch t deviates by more than a few years from the epoch T (i.e. from the epoch $T_H = 1991.25$ for Hipparcos stars).

There are many, more complicated applications of statistical astrometry. An example for such a problem is the combination (see Section 4.4) of an instantaneously measured catalogue (such as the Hipparcos Catalogue) with a mean catalogue, derived by averaging over observations obtained during a few centuries (such as the basic FK5). We refer to Wielen (1997).

4. IMPLICATIONS FOR USING HIPPARCOS RESULTS

We now discuss some important implications of the existence of cosmic errors in the proper motions and positions given in the Hipparcos Catalogue.

4.1. Bright Stars

For bright stars, such as the basic FK5 stars and hence probably for most of the 'survey stars' of Hipparcos, the cosmic errors are important and often much larger than the measuring errors. The cosmic errors should not be neglected in applications of Hipparcos data, especially in an error budget. Otherwise, the Hipparcos Catalogue may sometimes be blamed for having severely underestimated the measuring errors, while actually the effects of cosmic errors are seen.

4.2. Faint Stars

For stars fainter than about $V = 7$ mag, we have no direct observational estimate how large their cosmic errors are, mainly because we have no ground-based data of sufficient accuracy. We expect from theoretical considerations, however, that the cosmic errors are smaller for fainter stars than for bright stars. Together with the larger measuring errors of fainter stars, the cosmic errors play probably a less significant role for fainter stars.

4.3. Reference Frames

The Hipparcos stars will be often used as reference stars. Because of the random orientation of binary orbits in space and due to the random distribution of periastron passages in time, the cosmic errors in proper motions and positions should be randomly distributed in direction on the sky. Hence there should be no systematic errors in the reference frames caused by cosmic errors.

In order to keep the stochastic effect of cosmic errors on reference frames as small as possible, we should preferentially select those stars as reference stars which have the largest distances r from the Sun. This is because the cosmic errors decrease strongly with increasing distance r, as shown in Section 2.2. For all the Hipparcos stars we have the advantage that we can obtain a sufficiently accurate estimate of the distances by using the trigonometric parallaxes given in the Hipparcos Catalogue.

4.4. FK6 Catalogue

The combination of the Hipparcos results with the ground-based data contained in the basic FK5, will provide a new catalogue: the FK6. The proper motions and positions will be more accurate than those provided either by the Hipparcos Catalogue or by the FK5 alone. In the construction of the FK6, the cosmic errors will be properly taken into account. For most objects, the FK6 will give a variety of solutions: (1) a classical 'single-star mode', in which cosmic errors are neglected; (2) a 'short-term prediction' for epochs close to T_H of Hipparcos; (3) a 'long-term prediction' for epochs far from T_H; (4) special solutions for known binaries. The short-term prediction will closely follow the Hipparcos data, while the long-term prediction is governed by the FK5 data, because of the cosmic errors in the Hipparcos results. We shall also provide rules for a smooth transition from the short-term prediction to the long-term behaviour, based on the principles of statistical astrometry.

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