A RELATIVISTIC MODEL FOR THE GAIA OBSERVATIONS

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ABSTRACT

Exact general relativistic observation equations for the GAIA concept were derived for a *static* sphere, i.e., with only stellar angular coordinates as unknowns. These equations were then put to test in an end-to-end simulation of the current baseline mission. Although the applicability of the present model is limited to the case of an observer in the gravitational well of a spherical non-rotating Sun, the results show that, despite cumbersome formalism, measurements of angular distances among stars good to ~ 100 μarcsec (as expected for stars of $V = 16$ mag) can be modeled to yield estimates of the relativistic astrometric parameters with errors of ~ 20 μarcsec after only one year of satellite operations.

Key words: space astrometry (GAIA); methods: data analysis; relativity.

1. INTRODUCTION

The expected precision of the astrometric measurements of GAIA, $10-20\mu$arcsec down to $V = 16$ mag and possibly fainter (Lindegren & Perryman 1996), is such that significant systematic errors would remain if both observer and observables are not rigorously described. Therefore, it is mandatory that the GAIA observations be modelled as consistently as possible in a framework which utilizes an advanced theory of gravitation.

In Hipparcos the observations were pre-corrected for relativistic effects to $(v/c)^2 \simeq M/R_\oplus \sim 2$ mas, where $M = GM_\odot/c^2$ (i.e., half of the Schwarzschild radius) and $R_\oplus$ is the Sun-Earth distance. The observation equations were then formulated in flat (Euclidean) space.

At the level of accuracy at which we will pushed by the GAIA measurements, Epstein & Shapiro (1980) have shown that the post-post Newtonian (ppN) correction to light deflection due to the spherical Sun amounts to 12 μarcsec, that from the Sun quadrupole moment to $\simeq 0.2$ μarcsec, and that due to the Sun's rotation to $\simeq 0.7$ μarcsec; all these values are intended for photons grazing the solar limb. At 90° from the Sun the ppN contribution is down to 0.1 μarcsec.

More important for GAIA are the metric perturbations due to the solar system planets. Photons just grazing the limb of the giant planet Jupiter are deflected by $\sim$1.7 mas, and the presence of the Earth gravitational field amounts to ~ 6 μarcsec.

The are different startegies for taking into account relativistic effects when modeling accurate astrometric data. Brumberg (1991) and Klioner & Kopejkin (1992) use a post-Newtonian (perturbative) formulation of general relativity, while Soffel (1989) favours the PPN formulation, which would allow a more direct comparison among different theories of gravity.

We followed a non-perturbative approach after making the following simplifying assumptions:

(a) the space-time is only due to a spherical, non-rotating Sun; therefore we can use plain Schwarzschild metric;

(b) the GAIA observer is placed in a spatially circular orbit around the Sun at $R_\oplus$.

In this paper we first describe the logical path followed to derive the relativistic observation equations; we then follow with a brief account of the main results from our end-to-end simulations. Details of the mathematical formulation are given by de Felice et al. (1997).

2. BUILDING A GENERAL RELATIVISTIC MODEL

2.1. Observables

In principle, observables are the photons reaching GAIA (the observer) at the same proper time τ and from different directions across the visible sky. In particular, the *angle* between a star pair, i.e., between directions i and j is:

$$\cos\psi_{ij} = \frac{h_{\alpha\beta}k_i^\alpha k_j^\beta}{(h_{\iota\pi}k_i^\iota k_i^\pi)^{1/2}(h_{\rho\sigma}k_j^\rho k_j^\sigma)^{1/2}}, \qquad (1)$$

where k_i^α and k_j^β are the components of the tangents to the null geodesics of the photons emitted by the two stars, and $h_{\alpha\beta}$ is a tensor operator which projects in the rest frame of GAIA (in the Euclidean sense). The spatial components of the 4-vector k_i^α are, by definition, the derivatives:

$$k^r \equiv dr/dt\ ,\quad k^\phi \equiv d\phi/dt,\ and\ k^\theta \equiv d\theta/dt$$

where t is the coordinate time.

2.2. Observation Equations

The analytical expressions for the null geodesics in Equation 1 are given, for example, in de Felice & Clarke (1990). These are given in terms of the constants of motion of the photons, i.e., total energy (E), total angular momentum (L), and the azimuthal component of the angular momentum (l).

To derive the observation equations one needs to build relations among the constants of motion E, L, and l of the emitted photons and the Schwarzschild coordinates of the emitting stars (ϕ_i, θ_i, r_i), so that the star coordinates can be explicitely inserted in Equation 1.

This is where we used the assumption of a static sphere: all stars were supposed to lie sufficiently far from the Sun for the approximation $r_i \sim \infty$ to hold. Then, the relations seeked for are derived by integrating the two equations:

$$\int_\infty^{r_o} dr/\dot{r} = \int_{\theta_i}^{\pi/2} d\theta/\dot{\theta} \qquad (2)$$

and:

$$\int_{\theta_i}^{\pi/2} d\theta/\dot{\theta} = \int_{\phi_i}^{\phi_o} d\phi/\dot{\phi} \qquad (3)$$

where $\dot{r} \equiv k^r$, $\dot{\theta} \equiv k^\theta$, etc., and ϕ_o and r_o are the coordinates of the observer ($\theta_o = \pi/2$) at the (proper) time of the observation.

3. THE END-TO-END SIMULATION

The first step in the developement of an end-to-end simulation is to write Equation 1 in an explicit form relating the measurements to the unknown quantities (ϕ_i, θ_i), and this was accomplished in the way described in the previous section.

We then need to adopt a statistical tool to *estimate* the Schwarzschild coordinates of the stars from the measured arcs. For this, we decided to utilize linear least-squares which, as observation equations, take the linearized version of $cos\psi_{ij}$, i.e.

$$-\sin\psi_{ij}\,\delta\psi_{ij} = A_i\,\delta\theta_i + B_i\,\delta\phi_i + A_j\,\delta\theta_j + B_j\,\delta\phi_j, \quad (4)$$

where $\delta\psi_{ij}$ is the difference between the observed, highly accurate, angle and some a priori knowledge of it derived from approximate values of the Schwarzschild coordinates and, therefore, of the constants of motion of the associated photons. Correspondingly, the small differences $\delta\phi$ and $\delta\theta$ are the corrections to the approximate (catalogue) values one wishes to deduce from the least-squares estimation procedure (see 3.1.). The coefficients A_i, B_i, etc., can be derived analytically (de Felice et al. 1997) [1] and are calculated, likewise $sin\psi_{ij}$, from the catalogue values of the coordinates.

3.1. Simulating Relativistic Observations

The simulation code is an adaptation of that used in Lattanzi et al. (1991; see also Galligani et al. 1989) for the assessment of the astrometric accuracy of the sphere reconstruction in the Hipparcos mission; the most relevant changes concerned the calculations of the relativistic quantities, both observations and coefficients of the linearized observation equations.

We generate lists of 2000 stars randomly distributed on the (not directly observable) *Schwarzschild sky*, i.e., their positions on the Schwarzschild sphere are uniquely specified using azimuth (ϕ) and colatitude (θ) only. Perspective effects (parallax) and effects due to relative motions thoughgout the Galaxy (proper motions) were not considered. Also, stars were all assigned the same survey limiting magnitude V=16. The star positions derived from the random number generators are named *true* positions. From these, approximate locations, called *catalogue positions*, are generated by perturbing the true values of 2 mas. Such a *catalogue* simulates our best guess for where the stars are on the sky before the GAIA measurements. Catalogue values are used to compute the coefficients of the linearized equations and the approximate (catalogue) values of the angles among star pairs observed by the satellite needed to calculate $sin\psi$ and $\delta\psi$ on the left-hand side of Equation 4.

The satellite is made to sweep the sky according to a given scanning law quite similar to that successfully implemented on Hipparcos; the spin axis precesses around the Sun at a rate of ~ 6.4 *revolution/year* and with a constant angle of 43°. Table 1 lists some of the parameters of the reference mission used in this simulation. Stars that at any given time are

[1] We stress that such linearization of Equation 1 is not an approximation imposed on our derivations; it is just a necessity of the least-squares estimation procedure utilized.

Table 1. Most relevant parameters of the GAIA simulation. Notice that single-measurement error *refers to the error of one measured angle between two generic stars as observed during one revolution around the spin axis.*

parameter	numerical value	comment
orbital radius (r_o)	1.496×10^{11} m	same as Earth's orbital radius ($R_\oplus$)
precession angle	43°	same as solar aspect angle
satellite spin period	128 min	
angles between interferometers	54° (I1-I2), 78°.5 (I2-I3)	
coherent field of each interferometer	1.6°	
mission duration	1 year	static simulation
No. of simulated stars	2000	
No. of unknowns	4000	only Schwarzschild coordinates estimated
Catalogue error	2 mas	error of initial values in linearized observation equations
Single-measurement error	100 μarcsec	as expected for V∼ 16 stars

'seen' within a strip 1.6° wide along the great circle being scanned are considered observable and are given further consideration; a great circle (one revolution) is completed in about 2.1 hours. From the true coordinates of the stars in the strip we derive the constants of motions of the emitted photons and from these, utilizing Equation 1 directly, the *true* angles between each possible distinct pair that can be formed with the visible stars. Of these angles, only those satisfying the relations $\psi_{ij} = 54° \pm 1.6°$ or $\psi_{ij} = 78.5° \pm 1.6°$ are retained as *observed.* These observability conditions are the result of current expectations for the GAIA payload. The optical configuration of the present baseline mission (Lindegren & Perryman 1996) shows three stacked identical interferometers (I1, I2, and I3) each with a large field of view and pointing toward widely separated directions at angles $\sim 60°$ (I1-I2) and $\sim 80°$ (I2-I3) (Table 1).

Finally, for this investigation it is sufficient to say that the GAIA detection system is capable of measuring those wide angles with an error of ~ 100 μarcsec when *both* stars are of magnitude $V = 16$ mag. This is the error that is added to the true angles which passed the observability conditions for the generation of the *observed* angles to be used in the observation equations.

The simulation of one year of uninterrupted observations yielded 78050 observed arcs which were then used for the estimation of 4000 (two coordinates for each of the 2000 simulated stars) stellar unknowns. Each star has, on average, ∼40 connections; however, the scanning law favours stars at colatidudes near 43° and 137° with more than twice as many observations as the stars near the 'equator' ($\theta \sim 90°$).

4. RESULTS AND CONCLUSIONS

The numerical code used for the solution of the system of equations resulting from the simulations just described is, a part for minor changes to update it to faster machines, the one described in Galligani et al. (1989) and utilized on simulated as well as real Hipparcos data. The code implements an iterative algorithm which uses the conjugate-gradient method to solve, in the least-squares sense, large and sparse overdetermined systems like ours (the number of non zero elements represents $\simeq 0.1\%$ of the total).

This iterative method has also a 'built-in' way of dealing with rank-deficient matrices. In the presence of rank deficiency the solution returned by the program is that of *minimum norm*, i.e., among the infinite m-dimensional vectors (m is the number of rows of the desing matrix) satisfying the system of m equations, that with the smallest modulus is chosen.

Table 2. Mean and standard deviation of the true errors of the least-squares adjustment.

	μ (μarcsec)	σ (μarcsec)	No. of stars
$\delta\hat{\phi}$	< 1 μarcsec	36	1971 (98.6%)
$\delta\hat{\theta}$	< 1 μarcsec	21	1986 (99.3%)

Table 2 show the results of a typical sphere adjustment run on a set of simulated data.

The individual differences used to compute the average values in Table 2 are the *true* errors defined as $\delta\hat{\phi}_i = \phi_i(true) - \phi_i(adj)$ (and similarly for $\delta\hat{\theta}_i$), where $\phi_i(true)$ is the true Schwarzschild azimuth of star i and $\phi_i(adj)$ is the value derived from the results of the least-squares adjustment; it is:

$$\phi_i(adj) = \delta\tilde{\phi}_i + \tilde{\phi}_i$$

where $\delta\tilde{\phi}_i$ is the least-sqares estimate for star i and $\tilde{\phi}_i$ the corresponding initial, *catalogue*, value used in the calculation of the coefficients of the linearized equations.

The third column of Table 2 shows that there were few outliers that were removed before calculating the average errors. These are typically stars with a critically low number of connections.

In the absence of correlations among observations one would anticipate average errors on $\phi(adj)$ and $\theta(adj)$ of the order of $\frac{\sqrt{2}\ 100}{\sqrt{40}} \simeq 23$ μarcsec, where the factor 100 μarcsec is the single-measurement error and the value 40 is the typical average number of connections per stars (see previous section). On the other hand, because of the geometry of the arcs observed by GAIA, the ϕ error is expected to be larger than the θ error. From the values in Table 2 we derive the empirical value $\sigma_\phi/\sigma_\theta \simeq 1.7$. This is in quite good agreement with the theoretical prediction $\sigma_\phi/\sigma_\theta \simeq 1.6$ derived by Betti and Sanso' (1985) in similar circumstances but in the context of the Hipparcos mission.

The very small values obtained for μ_ϕ and μ_θ suggest that the reconstructed Schwarzschild sphere defines, on average, a set of spherical coordinates very close to the original, true, set. In a Newtonian framework, we would have said that the reconstructed sphere defines a coordinate system closely related to that of the true positions, thus pointing to the absence of overall residual distortions in the reconstructed sphere.

Despite the complications of the model and its limitations, the principles of global astrometry successfully implemented with Hipparcos appear to work in a relativistic scenario. We should therefore be able to fully exploit the precision promised by GAIA for $\sim$10 μarcsec astrometry.

We anticipate that, if the spherical non-rotating Sun is retained as the only cause for the space-time curvature, fully dynamical equations (i.e., including perspective effects – parallax – and intrinsic motions of the stars -$d\phi_i/dt$ and $d\theta_i/dt$) can be derived in a way similar to that described here.

Future work will address the complications of a more general expression for the adopted metric.

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