LARGE-SCALE DISTORTIONS OF HIPPARCOS-LIKE COORDINATE SYSTEMS

V.V. Makarov
Copenhagen University Observatory, Juliane Maries Vej 30, 2100 Copenhagen Ø, Denmark

ABSTRACT

The problem of accidental error propagation in the most general 'global solution' of a Hipparcos–like mission for coordinates of stars is addressed. Unlike the one–dimensional intermediate 'star–abscissae' solution, which has actually been used in the Hipparcos reductions, the global solution is free of wild amplification of random errors at certain harmonics, thanks to the intrinsic smoothing capability of angular measurements on the sphere. The choice of a basic angle is therefore not a critical issue for a future satellite. The astrometric reference frame is expected to be rigid with any basic angle in the range of 30° to 150°, as demonstrated by means of a spherical harmonic technique. A basic angle of 90° would be a good choice, for example.

Key words: space astrometry; reduction techniques.

1. INTRODUCTION

Following the remarkable success of the ESA Hipparcos mission, several proposals for a future astrometric satellite are being discussed at present. One of them, called GAIA (Lindegren & Perryman 1995), obtained a broad support of the astronomical community due to excellent prospects in various fields of astrophysics and astrometry. Some interesting smaller national projects, such as Struve (Yershov et al. 1995) and DIVA (Röser et al. 1997) can also be mentioned. All these projects are based on the same observational principle, which proved to be highly efficient in the Hipparcos mission.

The observations of a scanning astrometry satellite of the Hipparcos type are performed by a telescope with two fields separated by a 'basic angle' on the sky, but superposed in the focal plane in an area of about one square degree. The basic angle is physically implemented by a 'beam combiner' sending the light from the two fields into a single telescope. The beam combiner consists of two plane mirrors bonded to each other at an angle equal to half the basic angle, which therefore in fact remains very constant. The satellite spins slowly, with a period of revolution of about two hours, so that the two fields scan approximately along a great circle. Accurate astrometric observation is performed only in the direction of the great circle, not perpendicularly to the circle.

The complete set of observations during the mission can be thought of as a great number of arcs, connecting pairs of programme stars. The length of each arc is close to the value of the basic angle, within the width of the field of view of the telescope. The core of the global solution idea is that this set of one-dimensional measurements along great circles can be directly tied up into a coherent system of two-dimensional coordinates on the sphere, by a least-squares method, for example. The method was tested in numerical simulations by Bucciarelli et al. (1991), and it was adopted as an alternative for the Hipparcos data reductions, although never implemented in practice. It should be noted that the consideration of the global solution method in the present paper is vastly simplified compared to any practical implementation. For example, we do not consider at all the intervening tasks of the instrumental calibration and attitude parameters determination.

In the baseline method, which was actually implemented for Hipparcos, an intermediate step of 'great circle reductions' is used (Lindegren & Kovalevsky 1989). In this approach, the set of observations from about five consecutive revolutions is first treated in a great circle reduction providing an estimate of each star's 'abscissa', i.e., the projection of its position onto a fixed 'reference great circle'. The solution provides in addition several instrument parameters such as the basic angle, field rotation and scale value, which are assumed to remain constant during an observation set. In a later stage of the astrometric data processing the abscissae are combined to yield, among other quantities, the celestial coordinates α and δ of the stars.

Bucciarelli et al. (1991) demonstrated that the global solution is slightly superior to the baseline method with respect to the final precision of the resulting astrometric catalogue. It will undoubtedly be considered as the basic option for a future astrometric mission. This paper is a first attempt at an analytical study of the error propagation in the global solution technique. The question addressed here is whether the strategy of observations with a large basic angle and a relatively small field of view, coupled with a global solution approach, matches the high astrometric requirements of a future interferometric mission.

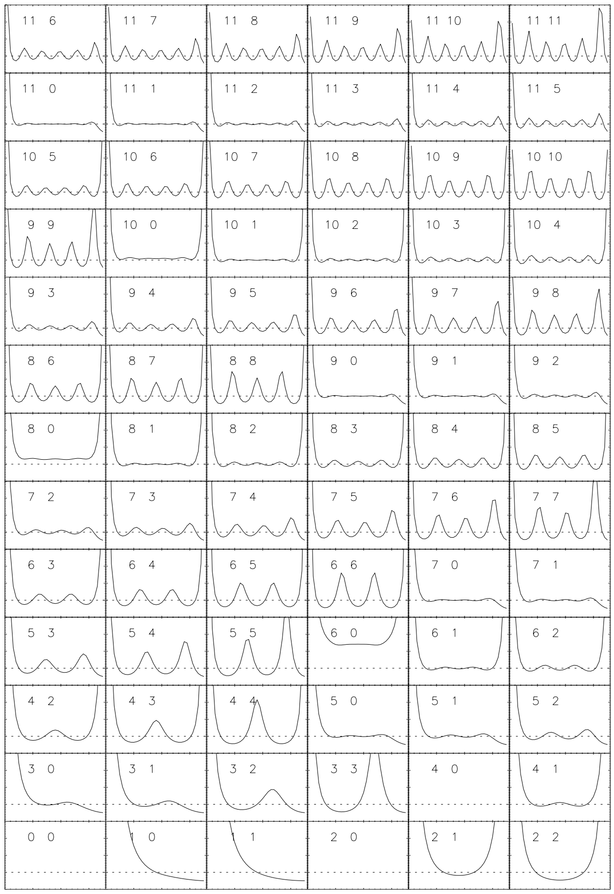

Figure 1. Power spectrum coefficients λP_j for $\Delta\alpha \cos\delta$ unknowns as functions of basic angle. The range of each plot is 0° to 180° horizontally and 0 to 4 vertically. The indices of spherical functions are given inside each plot.

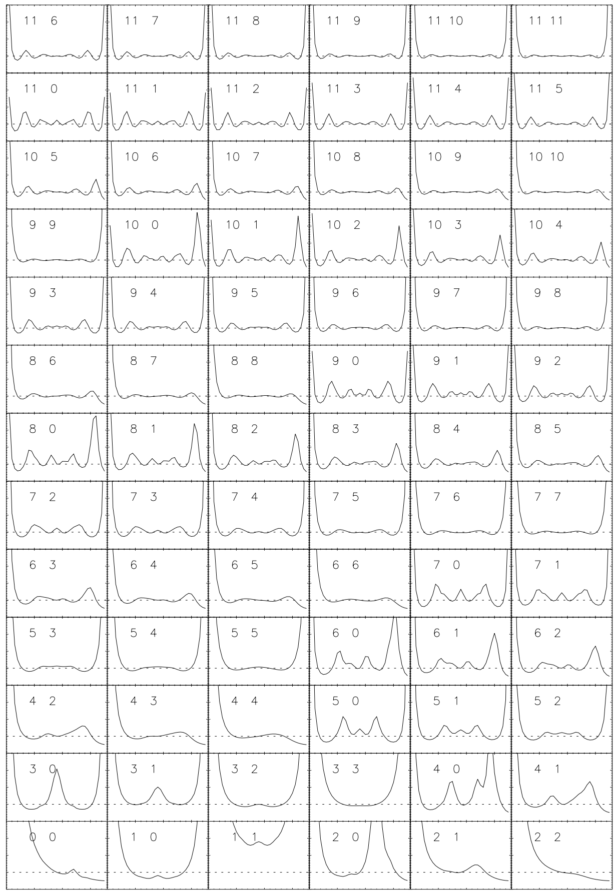

Figure 2. The same as Figure 1, but for $\Delta\delta$.

2. SPHERICAL ORTHOGONAL FUNCTIONS

A rather conventional technique of spherical function representation of astrometric errors is used in this paper. The (small) errors of angular coordinates are represented by the expansion:

$$\Delta\alpha \cos\delta = \sum_{j=0}^{J} a_j Y_j(\alpha, \delta) + \epsilon \qquad (1)$$

$$\Delta\delta = \sum_{j=0}^{J} b_j Y_j(\alpha, \delta) + \epsilon \qquad (2)$$

where Y_j are spherical orthogonal functions, Δc is conceived as the difference between the observed coordinate c_o and the true coordinate c_t, and ϵ is the high-frequency component of the noise. The spherical functions are related to associated Legendre polynomials by the equations

$$Y_{nms} = R_{nm} P_{nm}(\cos\delta) \sin m\alpha$$
$$Y_{nmc} = R_{nm} P_{nm}(\cos\delta) \cos m\alpha$$

The index j counts all different spherical orthogonal functions from 0 to $J = (N+1)^2$.

In this application, we have to consider a discrete set of points (target stars) on the sphere, for which the corrections $(\Delta\alpha \cos\delta, \Delta\delta)$ to the input catalogue positions should be determined. The condition of orthogonality in the discrete case is:

$$\sum_{i \in \Omega} Y_j(\alpha_i, \delta_i) Y_l(\alpha_i, \delta_i) = I \cdot \delta_{jl} \qquad (3)$$

where Ω is the discrete set of points, and I is the total number of points, provided the stars are uniformly distributed over the sky.

When the number of terms J in Equation 1 is close to the number of stars I, the representation by the spherical functions is fairly complete, and the high-frequency term ϵ is relatively small. The average standard error over the sky can then be calculated, for δ, for example, as:

$$\bar{\sigma}_\delta^2 \approx \sum_{j=0}^{J} \mathrm{var}[b_j] \qquad (4)$$

The covariances do not enter this expression, due to the orthogonality of the basic functions.

The numbers $P_j = \mathrm{var}[b_j]/\sigma_0^2$, where σ_0^2 is the variance of an elementary observation, can be interpreted as power spectrum of the random error, in analogy with the Fourier power spectrum of star abscissae errors, derived in Makarov et al. (1995). The latter depends critically on the value of the basic angle γ. As was shown by Makarov (1992), wild amplifications of abscissae errors at certain harmonics take place, when γ is a simple fraction of 360° (like $\frac{1}{4}$ or $\frac{2}{5}$), which badly affects the overall astrometric precision (cf. Figure 2.1 in Perryman & Hassan 1989).

The power spectrum P_j describes how the coordinate variance is divided among the spherical harmonics. It shows also an expected character of accidental distortions of the resulting coordinate system. If the power spectrum is dominated by one or several specific terms, then a specific pattern of accidental distortions should be expected.

Given the relativeness of all measurements, there is no *a priori* way in which the orientation of the celestial coordinate axes can be derived purely from the space observations. The determination of the celestial positions therefore suffers from a singularity corresponding to the undefined orientation of the axes. Yet it can be shown that the degeneracy of the solution emerges only at certain spherical harmonics (Vityazev 1994). It makes therefore sense to estimate the precision of α, δ from the unaffected spectral terms, leaving the determination of the axes orientation as quite a separate problem.

3. NORMAL EQUATIONS

Under the above-mentioned simplifying assumptions, an observation equation follows directly from the geometry of an elementary observation. The fact is disregarded here, that not just a single pair of stars, but rather all stars within the two simultaneous fields of view are bridged in one observation. In fact, the distance between two simultaneously observed stars can differ a little from the value of basic angle γ. Besides, there is always some averaging of photon noise errors on the scale of the field of view size, that brings about additional smoothing at high-order accidental harmonics. It is intuitively clear, and can be proven by numerical simulations that the overall astrometric performance benefits a lot from a wide field of view. These effects are, however, not relevant to the purpose of this paper.

In the small-angle approximation, a linearized observation equation for a given pair of objects (p,q) is written as:

$$\Delta d = \Delta\alpha^*_p \sin\phi_p + \Delta\alpha^*_q \sin\phi_q - \Delta\delta_p \cos\phi_p - \Delta\delta_q \cos\phi_q \qquad (5)$$

where Δd is the correction to the pre-calculated distance $d \approx \gamma$, and ϕ_r, $r = p,q$ is the position angle of the scan in the direction to the other object of the pair. Using Equations 1 and 2, the observation equation can be rewritten in terms of spherical functions as:

$$\Delta d = \sum_{j=0}^{J} (a_j[Y_j(p)\sin\phi_p + Y_j(q)\sin\phi_q] - b_j[Y_j(p)\cos\phi_p + Y_j(q)\cos\phi_q]) \qquad (6)$$

Thus, the $2I$ unknown corrections to angular coordinates are replaced by $2J$ unknown coefficients (a_j, b_j), provided $J \le I$. The set Λ of elementary observation equations can be solved by a least-squares method. The normal matrix $\mathbf{N}$ can be evaluated for (a_j, b_j) as unknowns. This matrix turns out to be very sparse and relatively easy to evaluate. For example, the off–diagonal elements, corresponding to covariances between the $a-$ and $b-$terms, are always equal to zero. This allows us to split the problem into two independent smaller problems, for the two coordinates separately.

The same applies for any combinations of the $\sin -$ and $\cos -$ terms of tesseral and sectorial functions Y_{nm} $(m \neq 0)$. Moreover, all off–diagonal elements vanish, unless the order of the corresponding functions $Y_{n_1m_1}$ and $Y_{n_2m_2}$ is the same, $m_1 = m_2$, and the sum of their degrees $n_1 + n_2$ is even.

The diagonal element (j,j) for α, for example, is:

$$N^\alpha_{jj} = \sum_{(p,q)\in\Lambda} (Y_j^2(p)\sin^2\phi_p + Y_j^2(q)\sin^2\phi_q + 2\,Y_j(p)Y_j(q)\sin\phi_p\sin\phi_q) = \frac{1}{2}N_{\rm obs}\cdot I + 2\sum_{(p,q)\in\Lambda} Y_j(p)Y_j(q)\sin\phi_p\sin\phi_q$$

where $N_{\rm obs}$ is the number of elementary observations per star. The latter equality holds, provided the elementary observations are uniformly distributed on position angle ϕ for each star.

If the second term in the above equation, as well as all the off–diagonal elements, were equal to zero, the accidental error would be uniformly distributed among the spherical harmonics, and the coordinate error propagation would follow the $1/\sqrt{N_{\rm obs}}$ law. However, this is generally not the case.

The normal matrix was evaluated in the following way. The entire sphere was divided into 4584 regions of about 9 square degree each. A reference (primary) point was placed in the centre of each cell. The orthogonality of the basic functions was numerically asserted to a relative precision better than 10^{-4}. Then, a set of 50 secondary 'stars' was considered, regularly situated along the small circle of radius γ centered on each primary point. Thus, the total number of simulated stars in these calculations (and the number of elementary observations) was $\lambda = 229\,200$. Normal matrices were then calculated for the 144 first terms of expansions (1) and (2), up to $Y_{11\,11}$, for values of γ in the range 5 to 175 degrees, and covariance matrices were evaluated. The resulting power spectrum coefficients λP_j are shown in Figures 1 and 2, as functions of γ.

All the curves in Figures 1 and 2 are fairly smooth, and this makes an important difference with the case of one–dimensional great–circle solution. There are no heavily dominating peaks in the spectra, notwithstanding the actually infinitely small field of view in this analysis. The largest variances are found for $Y^{(\alpha)}_{00}$, $Y^{(\alpha)}_{20}$ and $Y^{(\alpha)}_{40}$, where they are far beyond the range of the plots. The correlation coefficients between these terms are close to ± 1. This is most probably a signature of the rank–deficiency of the

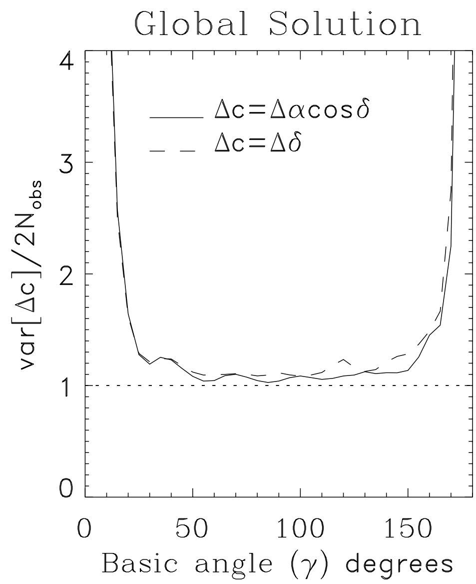

Figure 3. Estimated overall precision of coordinates, obtained in the global solution, as a function of basic angle.

problem. Rotation around the poles is in fact very similar to the distortion represented by $Y_{00}^{(\alpha)}$, a significant difference being only in the circumpolar regions, where the observation equation (5) becomes invalid. Moreover, such a rotation can be expressed through a certain linear combination of (completely correlated) $Y_{00}^{(\alpha)}$, $Y_{20}^{(\alpha)}$, $Y_{40}^{(\alpha)}$, ... zonal terms (Vityazev 1994). These terms should probably be fixed by the determination of the coordinate system orientation, which is quite a separate problem beyond the subject of this paper.

As could be expected, local maxima of λP_j are found for the sectorial harmonics in $\Delta\alpha\cos\delta$: $\lambda P_{33}^{(\alpha)} = 5.1$ at $\gamma = 120°$, $\lambda P_{44}^{(\alpha)} = 3.1$ at $\gamma = 90°$ and $\lambda P_{55}^{(\alpha)} = 2.4$ and 4.6 at $\gamma = 72°$ and 144°, respectively. As for the $\Delta\delta$ corrections, the most prominent maxima are $\lambda P_{20}^{(\delta)} = 8.5$ at $\gamma = 120°$ and $\lambda P_{40}^{(\delta)} = 4.8$ at $\gamma = 145°$.

In principle, with such a power spectrum, the general character of accidental distortions of the resulting coordinate system can be predicted. With a basic angle of 120°, for example, the accidental distortions would be most probably dominated by those represented by $Y_{20}^{(\delta)}$ and $Y_{33}^{(\alpha)}$. The amplitudes of these distortions can be predicted, but not the phase (for the $Y_{33}^{(\alpha)}$). Perhaps it is more interesting to estimate the overall astrometric performance, as a function of γ.

The overall precision for both coordinates is estimated by Equation 4. The result is shown in Figure 3, where the three poorly determined terms $Y_{00}^{(\alpha)}$, $Y_{20}^{(\alpha)}$ and $Y_{40}^{(\alpha)}$ have been excluded. It confirms that the astrometric quality of the global solution is virtually the same with any basic angle in the range 30° to 150°.

4. CONCLUSIONS

We have shown, that the expected amplitudes of large-scale distortions of Hipparcos-like coordinate systems can be evaluated without solving the observation equations, in the most general global solution. The attributed power spectrum of these distortions turns out to be fairly smooth, whichever value for the basic angle is chosen within the interval 30° to 150°.

This allows a free choice of the basic angle for a future astrometric mission. A basic angle of 90°, for example, would provide a coordinate system of high quality with respect to large-scale accidental distortions. Besides, it has important advantages as compared with the $\gamma = 58°$ of Hipparcos. Firstly, a larger basic angle brings better precision for parallaxes due to evident geometrical reasons. Secondly, a basic angle, which gives 360° by multiplication, provides a natural closure condition on each revolution of the satellite, greatly facilitating the self–calibration of the instrument, and somewhat relaxing requirements to its stability.

ACKNOWLEDGEMENTS

This work was supported by the Danish Space Board.

REFERENCES

Bucciarelli B., Lattanzi M.G., Migliaccio F., Sansó F., Sarasso M., 1991, Adv. Space Res., Vol. 11, No. 2, 79.

Lindegren L., Perryman M.A.C., 1995, The GAIA Concept, in Future Possibilities for Astrometry in Space, ESA SP-379, Cambridge, UK, p. 23

Lindegren L., Kovalevsky J., 1989, Overview of the Data Acquisition and Reductions, in The Hipparcos Mission, ESA SP–1111, 1

Makarov V.V., 1992, Pis'ma Astron. Zh., 18, 630 (Sov. Astron. Lett., 1992, 18(4), 252).

Makarov V.V., Høg E., Lindegren L., 1995, Experimental Astronomy, 6, 211.

Perryman M.A.C., Hassan H., 1989, *The Hipparcos mission*, Vol. 1, *The Hipparcos Satellite*. ESA SP–1111.

Röser S., Bastian U., deBoer K.S., Høg E., Roeser H.P., Schalinski C., Schilbach E., deVegt Ch., Wagner S., 1997, ESA SP–402, this volume

Vityazev V.V., 1994, Astron. Astrophys. Transactions, 4, 195.

Yershov V.N., Chubey M.S., Il'in A.E., Kopylov I.M., Gorshanov D.L., Kanayev I.I., Kirian T.R., 1995, STRUVE. Scientific grounds of the project (in Russian). Main Astronomical Observatory, SPb.