Update on inversion methodologies

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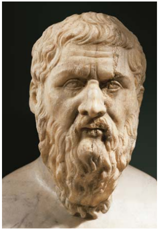

Introduction

PLATO mission

Scientific goals:

- study exoplanetary systems as a whole
  - mass, radius, age determination of host stars
- gain a better understanding of stars

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Asteroseismology is extremely important because of its ability to probe stellar interiors and its high precision

Asteroseismology

Direct approach

- **description**: search for optimal model in a restricted parameter space
- **advantages**: simplicity, physically coherent models

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- **advantages**: extracts more information from frequencies, open to new physics

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Inversion methods

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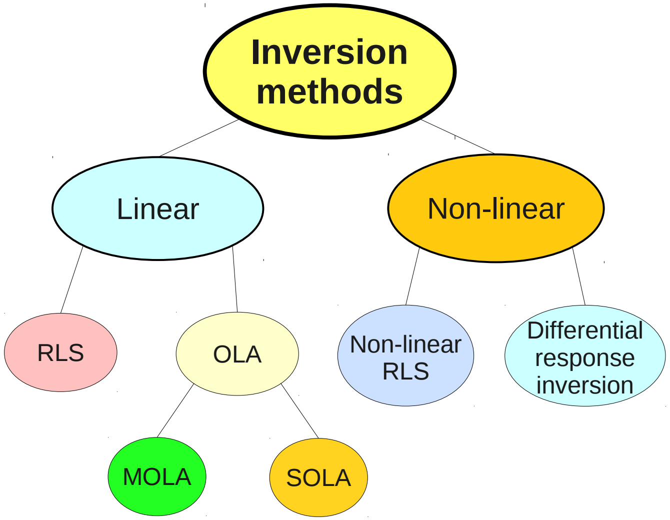
Comparison

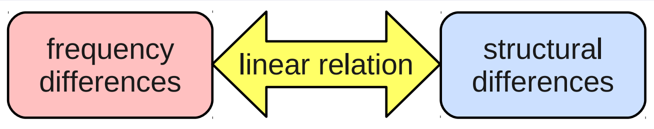
- rather than opposing each other, the two approaches are complementary:
  - the direct approach can provide an initial model for an inverse method

Classification of inversion methods



Linear inversion methods

- **assumption**: reference model sufficiently close to real star so that:



- **Rotation profile**: $\dfrac{\nu_{n\ell m} - \nu_{n\ell 0}}{m} = \displaystyle\int_0^R K_\Omega^{n\ell}(r)\Omega(r)dr$
- **Structural change**: $\dfrac{\delta\nu}{\nu} = \displaystyle\int_0^R \left[K^{n\ell}_{c^2\,\rho}(r)\frac{\delta c^2}{c^2} + K^{n\ell}_{\rho\,c^2}\frac{\delta\rho}{\rho}\right] dr$
  - the *kernels* $K_\Omega^{n\ell}$, $K^{n\ell}_{c^2\,\rho}$, and $K^{n\ell}_{\rho\,c^2}$ are deduced from the variational principle
- **Goal**: inverse above integral relations

Linear inversion methods

Averaging and cross-term kernels

- linear inversion $\Rightarrow$ solution is a linear combination of frequency differences:

$$\Omega_{\rm inv}(r_0) = \sum_{n\ell} c_{n\ell}(r_0)\frac{\nu_{n\ell m}-\nu_{n\ell 0}}{m} = \int_0^R \underbrace{\left[\sum_{n\ell} c_{n\ell}(r_0)K_\Omega^{n\ell}(r)\right]}_{\mathcal{K}_{\rm avg}(r_0,r)} \Omega(r)dr$$

$$\frac{\delta c_{\rm inv}^2(r_0)}{c^2(r_0)} = \sum_l c_l \frac{\delta\nu_l}{\nu_l} = \int_0^R \underbrace{\sum_l c_l K^l_{c^2,\rho}}_{\mathcal{K}_{\rm avg}(r_0,r)} \frac{\delta c^2}{c^2}dr + \int_0^R \underbrace{\sum_l c_l K^l_{\rho,c^2}}_{\mathcal{K}_{\rm cross}(r_0,r)} \frac{\delta\rho}{\rho}dr$$

- $\mathcal{K}_{\rm avg}$ = averaging kernel (*e.g.* Christensen-Dalsgaard et al., 1990)
- $\mathcal{K}_{\rm cross}$ = cross-term kernel

Linear inversion methods

Error propagation

- the observational errors propagate into the result as follows:

$$\sigma_{f(r_0)} = \sqrt{\sum_l c_l^2 \sigma_l^2}$$

- other sources of error are **not** taken into account in this formula:
  - poorly localised averaging kernel
  - strong cross-term kernel
  - underlying model too far from star

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Two approaches

- **RLS**: Regularised Least-Squares
- **OLA**: Optimally Localised Averages

The RLS method

- **RLS**: Regularised Least-Squares
- **Goal**: minimise frequency differences by adjusting internal profiles
- minimisation of following cost function:

$$J(f) = \underbrace{\sum_{l=1}^{L} \left\{ \frac{\frac{\nu_m^l - \nu_0^l}{m} - \int_0^R K_\Omega^l(r) f(r) dr}{\sigma_l} \right\}^2}_{\text{fit data}} + \underbrace{\Lambda \left\langle \frac{1}{\sigma^2} \right\rangle \int_0^R \left\{ \frac{d^2 f}{dr^2} \right\}^2 dr}_{\text{regularisation}}$$

- where:

$$\begin{aligned} f &= \sum_i a_i \phi_i(r) = \text{the function we're trying to invert} \\ \sigma_l &= \text{observational errors} \\ \Lambda &= \text{regularisation trade-off parameter} \end{aligned}$$

The OLA methods

- **OLA**: Optimally Localised Averages
- **Goal**: optimise the averaging kernels

- **MOLA**: Multiplicative OLA (Backus & Gilbert, 1968)

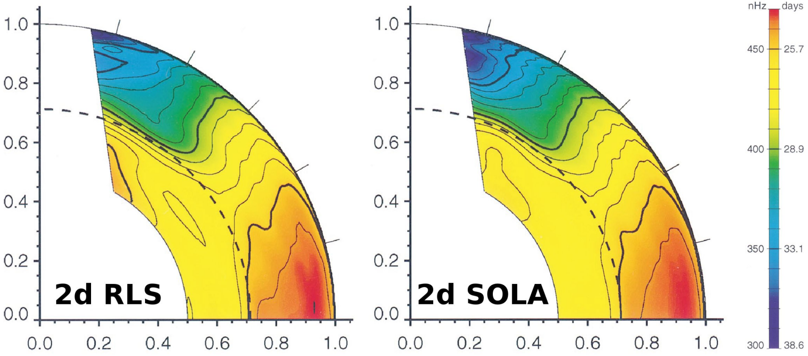
$$J(c_l(r_0)) = \underbrace{\int_0^R J(r_0,r)\,[\mathcal{K}_{\mathrm{avg}}(r_0,r)]^2 dr}_{\text{fit data}} + \underbrace{\frac{\tan\theta}{\langle\sigma^2\rangle}\sum_{l=1}^{L}(c_l\sigma_l)^2}_{\text{regularisation}} + \underbrace{\lambda\left\{1-\int_0^R \mathcal{K}_{\mathrm{avg}}\right\}}_{\mathcal{K}_{\mathrm{avg}}\ \text{unimodular}}$$

- **SOLA**: Subractive OLA (Pijpers & Thompson, 1992, 1994)

$$J(c_l(r_0)) = \int_0^R \Big[\underbrace{\mathcal{T}(r_0,r)}_{\text{target}} - \mathcal{K}_{\mathrm{avg}}(r_0,r)\Big]^2 dr + \ldots$$

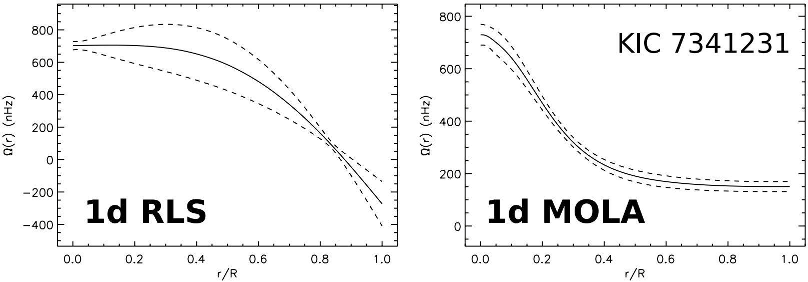
- MOLA method: fewer free parameters
- SOLA method: less computationally expensive (1 matrix inversion)

Some examples



The solar rotation profile
(Schou et al., 1998, see also Thompson et al., 2003)

Some examples



Rotation profile of a lower-giant-branch star
(Deheuvels et al., 2012)

Stellar parameters

- structural inversions are difficult for stars other than the Sun, due to the limited number of modes (*e.g.* Basu et al. 2002)
- one strategy is to invert stellar parameters rather than structural profiles

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How does it work?

$$\frac{\delta\rho_{\mathrm{inv}}(r_0)}{\rho(r_0)} = \int_0^R \mathcal{K}_{\mathrm{avg}}(r_0, r)\frac{\delta\rho}{\rho}dr + \int_0^R \mathcal{K}_{\mathrm{cross}}(r_0, r)\frac{\delta\Gamma_1}{\Gamma_1}dr$$

- an inversion gives you a weighted average of the underlying profile
- **idea**: directly search for the appropriate weighting which yields the stellar parameter
- carry out a SOLA inversion with a suitable target function:

$$\text{Target function} = \frac{4\pi r^2 \rho R}{M} \qquad \Rightarrow \qquad \text{stellar mean density}$$

Stellar parameters

What parameters are accessible?

- total angular momentum (Pijpers, 1998)
- mean density (Reese et al. 2012)
- acoustic radius (Buldgen, Master's thesis, 2013)
- age indicator, based on small frequency separation (Buldgen, Master's thesis, 2013)
- ...

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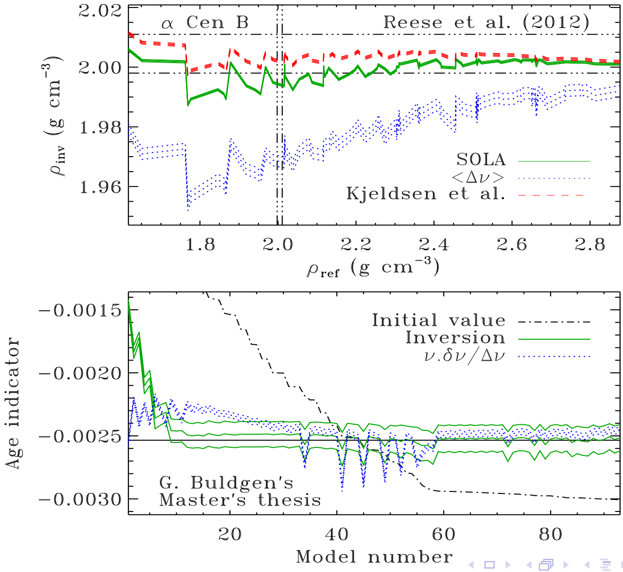
Non-linear extension

- **purpose**: extend the inversion's range of application
- **applies to**: mean density and acoustic radius
- optimally scaling the reference model before carrying out the inversion

Some examples



Non-linear inversion methods

- a second strategy for structural inversions in stars other than the Sun
- useful for stars with mixed modes which are highly sensitive to structural changes
- applies even when the reference model is far away from true structure

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Two approaches

- frequency-based approach
- approach based on internal phase-shifts

Non-linear RLS

Description

- iterated RLS inversions
- minimisation of the following cost function:

$$J(f) = \sum_i \left(\frac{\nu_i^{\mathrm{obs}} - \nu_i^{\mathrm{theo}}(f)}{\sigma_i} \right)^2 + \Lambda \left\{ \text{regularisation term} \right\}$$

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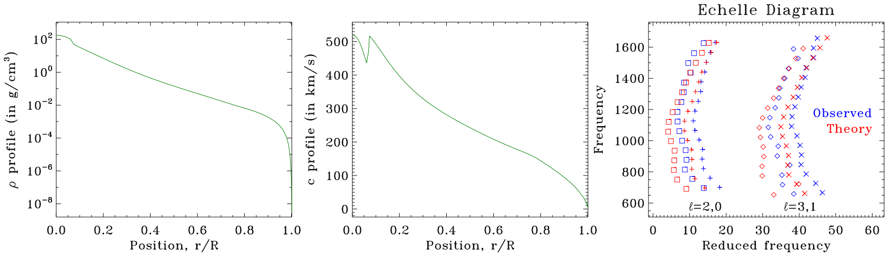
Different works

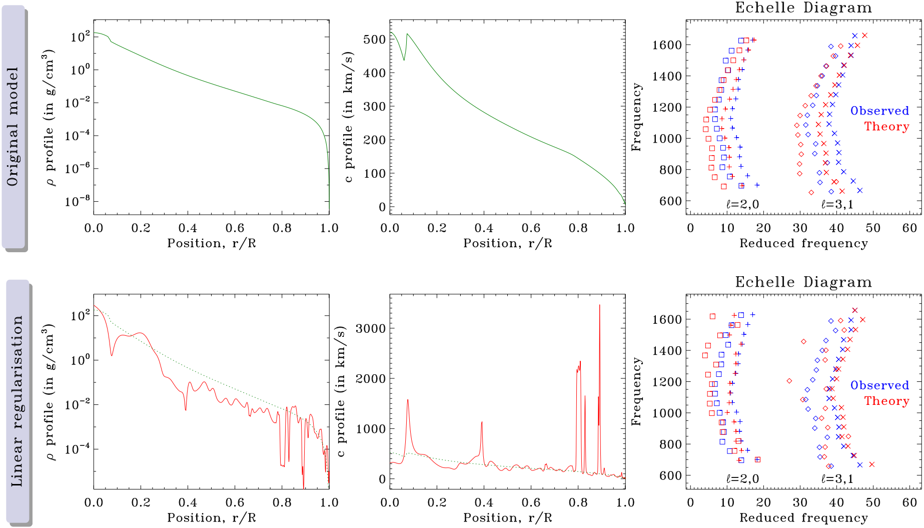
- **Antia (1996)**: inversion on (ρ, Γ_1), regularisation of $\left(\frac{\delta\rho}{\rho}, \frac{\delta\Gamma_1}{\Gamma_1}\right)$
- **Reese (ongoing)**: inversion on ρ, fixed Γ_1 profile, regularisation of ρ

Some examples

Original model



Some examples



Regularisation

- clearly, the regularisation term is unsuitable:

$$\Lambda \int_0^{R_{\mathrm{cut}}} \left(\frac{\partial^2 \rho}{\partial r^2}\right)^2 dr$$

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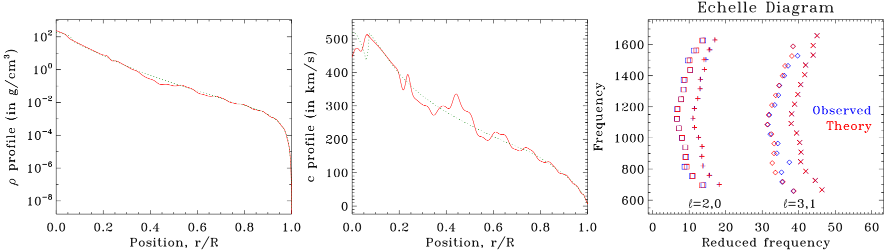
- therefore, a different regularisation term was tested out:

$$\Lambda \int_0^{R_{\mathrm{cut}}} \left(\frac{\partial^2 (\ln \rho)}{\partial r^2}\right)^2 dr$$

Some examples

Logarithmic regularisation



Some examples

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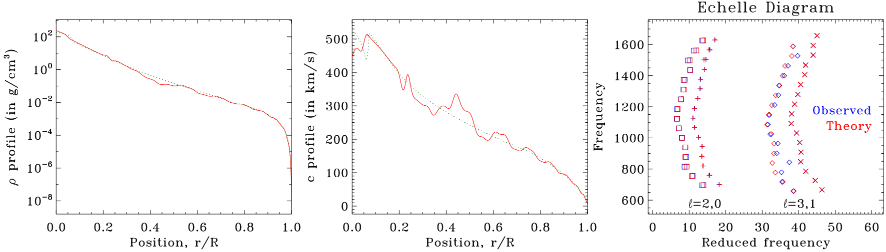

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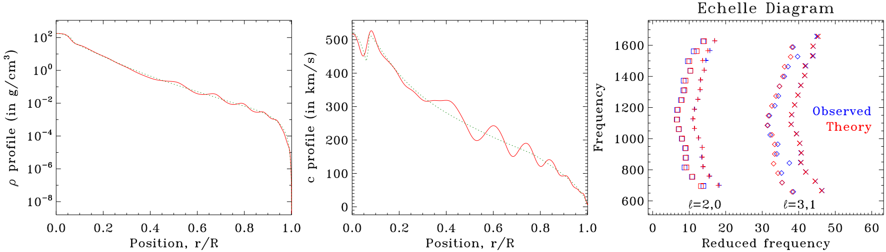
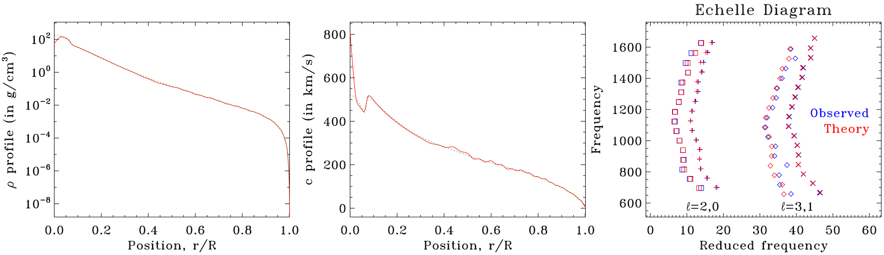


Some examples

With surface correction



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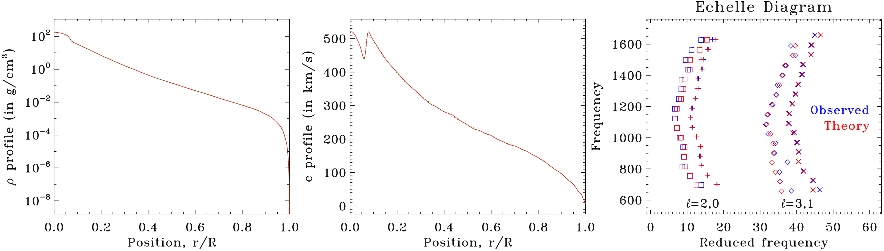

Some examples

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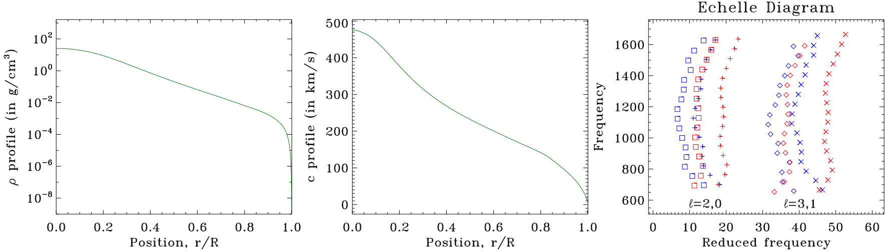

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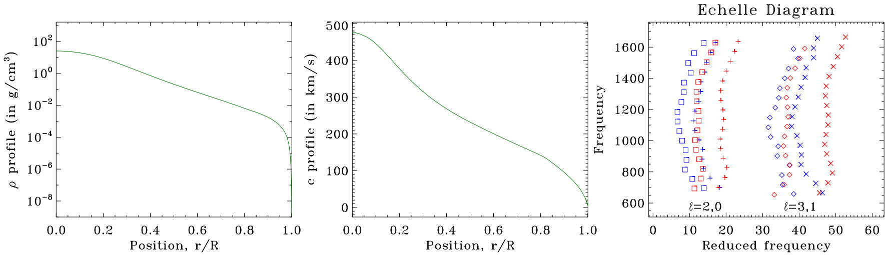

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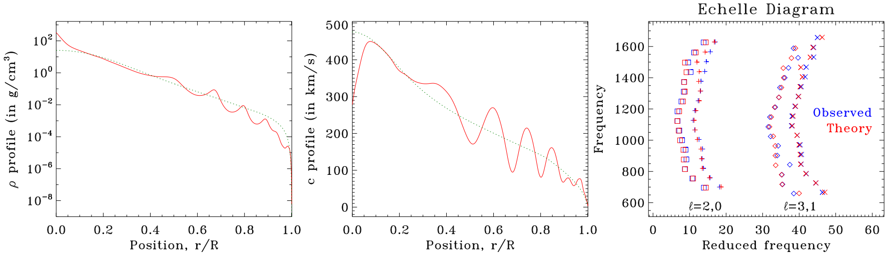


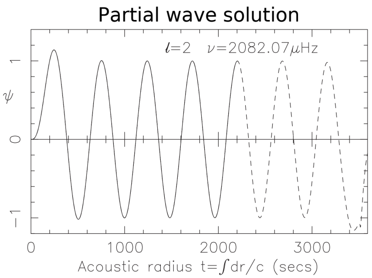
Differential Response Inversion

Description

1. Discretise (ρ, Γ_1) profiles up to a truncation point.
2. At the observed frequencies, obtain partial wave solutions and associated phase shifts.
3. Adjust model so that phase shifts become a function of frequency only.

Various articles

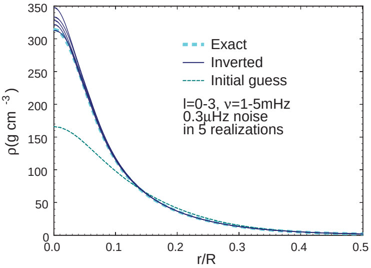
- Vorontsov (1998, 2001), Roxburgh (2002, 2010)



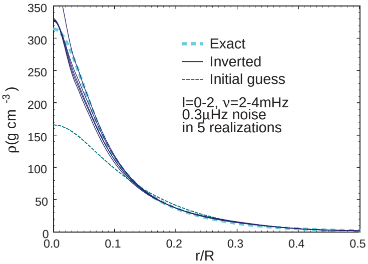
(Roxburgh & Vorontsov, 2003)

An example

Favourable



Unfavourable



(Roxburgh, 2002)

- multiple realisations are used to determine the uncertainty on the profile

Conclusion

- rotational inversions are feasible in stars other than the Sun
- structural inversions remain a challenge
- extraction of stellar parameters:
  - may provide a useful intermediate between scaling laws and inversions
  - still needs more exploration on large samples of models/stars
- non-linear inversions:
  - can reproduce frequency spectra
  - models can be somewhat unphysical
  - additional physics might improve results – room for new physical behaviours?